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RESEARCH PAPER

On Infinitesimal L_ω-smooth Functions.

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A B S T R A C T:

The aim of this paper is to study smoothness, approximate continuity, and approximate derivative in a nonstandard manner with respect to infinitesimal parameters. The new nonstandard introduced definitions are combined with standard and nonstandard intermediate value property. Particularly, we show that the existence $L_\omega -$ derivative of continuous and smooth function has the infinitesimal intermediate value property. Moreover, for the same result, we reduce the continuity condition to the infinitesimal intermediate value condition

KEY WORDS: infinitesimals, smooth functions, intermediate value property, continuity, symmetric functions.
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1.INTRODUCTION :

1.1. Nonstandard Analysis Background

During 1961-1966, A. Robinson developed a first rigorous foundation of nonstandard analysis (NSA) (Robinson, 1961; Robinson, 1996). He had been using theorems in mathematical logic in the holies to derive known mathematical results in a non-classical way. His method based on the theory of models for making an extension of the classical number systems by looking at nonstandard models of their respective theories. Infinitely small and infinitely large numbers were to be found in the enlargement ℝ of R. He was able to justify proofs using infinitesimals and that was not possible before his discovery. His article with Bernstein certainly showed that these methods were able to produce original solutions to unsolved mathematical questions as well (Robert, 1988). Throughout this paper, by ℝ, we mean the proper extension of conventional real numbers R, includes all real numbers together with nonstandard quantities.

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The elements of ℝ are often called hyperreals. In 1977 Nelson, E (Nelson, 1977) introduce a new approach to constructing NSA, depending on three basic principles named; Transfer, Idealization, Standardization. His approach known by Internal Set Theory (IST). In IST, every mathematical object is regards to be a set, and every set is standard and any set or formula is called internal in case it does not defend with the new predicate "standard" and its derivations. There are a several different approaches has been presented by other mathematicians for representing a Robinson's nonstandard analysis sense, one of the recent approach has been done by (Abdeljalil, 2018), he proposed a very simple method in practice to nonstandard analysis without using the ultrafilter. There were a number of studies have examined a results on nonstandard analysis and its application. A first nonstandard generalization of curvature and torsion and some other concepts in differential geometry had been presented by Hamad, I. (Hamad, Generalized curvature and torsion in nonstandard analysis, 2011). (Sun, 2015) were introduced an applications economics. Similarly (Duanmu, 2018) were applied the nonstandard analysis to Markov processes and statistical decision theory. An interesting application of a transfer principle for continuations of real functions to Levi-Civita field has been presented by (Bottazzi, 2018), (Ciurea, 2018) has constructed an approach for nonstandard analysis in a complete metric spaces (Jwahir, 2021) study convergent sequences via nonstandard transfer principle … etc.

The following are some basic definitions in nonstandard sense which are need in various places in this work.

A real number x is called limited if $|x| \leq r$ for some positive standard real numbers r, unlimited if $|x| > r$ for all positive standard real numbers r, infinitesimal if $|x| < r$ for all positive standard real numbers r, appreciable if it is limited not infinitesimal. Two real numbers x and y are said to be infinitely near if $x - y$ is infinitesimal, denoted by $x \cong y$. Every limited real number x is infinitely close to a unique standard number called the standard part or shadow of x. The monad of x, denoted by $m(x)$, is the set of all y such that $x \cong y$, by $m^{+}(0)$ we mean a positive part of infinitesimals. A standard function f is continuous at a standard point x_o if and only if $f(x) \cong f(x_o)$ for all $x \cong x_o$. It is s-continuous if and only if $f(x) \cong f(y)$ for all $x \cong y$. A subset E of $\mathbb{R}$ is closed if for all $x \in E$, there exist a standard x_o , $x_o \cong x$, then $x_o \in E$. An internal function $f: \mathbb{R} \to \mathbb{R}$ is said to have the infinitesimal intermediate value property (IIVP) provided that if a and b are real numbers such that $a < b$ and $f(a) \neq f(b)$, then for every λ between $f(a)$ and $f(b)$, $f(a) \lesssim \lambda \lesssim f(b)$, there exists a real number z ; $x \lesssim z \lesssim y$ such that $f(z) \cong \lambda$. A set $C \subseteq X$ is closed if and only if whenever $p \in X$ and $q \in C^*$ are such that $p \cong q$ then $p \in C$, where C^* is a nonstandard extension of C (Goldbring (2014), Corollary 9.10).

Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real (or complex) numbers. (i) If $a_n = 0$ for all standard n, there is an unlimited $\nu \in \mathbb{N}$ with $a_n = 0$ for all $n \leq \nu$. (ii) If $a_n \cong 0$ for all standard n, there is an illimited $\nu \in \mathbb{N}$ with $a_n \cong 0$ for all $n \leq \nu$ (Robert (1988), Robinson's Lemma).

For above definitions and other nonstandard background see (Nelson, 1977; Diener, F., & Diener, M., 1995; Goldblatt, 1998; Robinson, 1996; Hamad, 2016; Hamad & Hassan, 2021).

1.2. Standard Analysis Background

The concept of a smooth or differentiable function arises at several places in the applied mathematics. The application of smooth functions covers a vast range. Because smooth functions often constructed from smooth norms, many results discussed in this article concern the existence of differentiable norms. In particular, we shall characterize those functions that admit intermediate value property and is Baire1. The question that we seek for is of particular interest is the question concerning the role of nonstandard analysis for obtained distinguished and accurate new results about smooth functions. This fact will be noticeable during using nonstandard infinitesimal parameters. Many previous paper works have shown that approximation methods using different kinds of smoothness, for instance, see (Benyamini, Y. & Lindenstrauss, J., 2000; Borwein, J. M., & Preiss, D., 1987; Deville, R., Godefroy, G., & Zizler, V., 1993; Vanderwerff, 1992; Kiro, 2020).

Some classical standard definitions which are need are given in below:

A *Darboux* function is a real-valued function f which has the "*intermediate value property* (*IVP*)": Let $I \subset R$ be a nonempty interval and $f: I \subset R \longrightarrow R$ be a function. We say that f has the *intermediate value property* if for any $a, b \in I, a < b$ and any y between $f(a)$ and $f(b)$, there is some c between a and b with $f(c) = y$ (Ciesielski, 1997). A function $f: [0,1] \longrightarrow \mathbb{R}$ is *Baire*1 if and only if in any closed set C there is a point x_0 at which the restricted function $f|C$ is continuous. A function $f: [0,1] \longrightarrow \mathbb{R}$ is *Baire**1* if for every closed set C there is an open interval (a, b) with $(a, b) \cap C \neq \emptyset$ such that $f|C$ is continuous on (a, b) (O'Malley, 1976). A subset of a topological space is called *nowhere dense* or *rare* if its closure has empty interior. In a Metric space X we say that $A \subset X$ is *nowhere dense* if $\left(\overline{A}\right)^o = \emptyset$ or $A^c \supset G$ open and dense (Narici, L., & Beckenstein, E., 2011). A *Perfect set* is a closed set with no isolated points, every uncountable set E in a complete metric space is uncountable (Arkhangel'skii, 2001).

In NSA sense, Baire*1 can be define by: A function $f: [0,1] \longrightarrow \mathbb{R}$ is *Baire**1* if for every internal closed set C there is an internal open interval (a,b) with $(a,b) \in m(F)$ such that $f|C$ is continuous on (a,b).

This work draws formal links between classical concepts of smoothness, differentiability, approximate continuity, approximate derivative and nonstandard infinitesimals, s-continuity combine with IVP, IIVP, Baire1, and Baire*1.

2. MAIN RESULTS

We start this section with the first main definition and theorem combine between smoothness and Baire1 property of hyperreal valued internal functions.

Definition 2.1
Let $f: [a,b] \longrightarrow \mathbb{R}$ be an internal function. Then we say that f is an $L_\omega - smooth$ function if for each $x \in (a,b)$ and unlimited ω,

$$\left\{\frac{1}{\delta}\int_0^\delta |f(x+t)+f(x-t)-2f(x)|^\omega dt\right\}^\omega \cong 0 \quad \text{as} \quad \delta \cong 0.$$

Theorem 2.1
Let $f: [a,b] \longrightarrow \mathbb{R}$ be an internal $L_\omega - smooth$ function. Then f is Baire* 1.

Proof
Let $A_\omega = \{x \in \mathbb{R}\}$ such that : $\forall \varepsilon \in m^+(0)$,

$$\int_{x-\varepsilon}^{x+\varepsilon} |f(t)|^\omega dt \qquad \text{is unlimited.}$$

Then for all standard $x_0 \cong x$, we have that $x_0 \in A_\omega$. By Corollary 9.10 (Goldbring, (2014), we obtain A_ω is closed

Now, we try to show A_ω is countable, by contradiction assume that A_ω is uncountable then A_ω can be written as $A_\omega = P \cup \mathbb{N}^{st}$, where P is perfect and $\mathbb{N}^{st}$ is denote to the set of standard natural numbers and $P \cap \mathbb{N}^{st} = \emptyset$.

Let $(a,b) = P^c$ such that $b \in P \cap (0,1)$, then there is a positive infinitesimal δ such that

$$\int_a^{a+\delta} |f(b+t)+f(b-t)-2f(b)|^\omega dt < M,$$

for some constant M>0.

Let $\gamma \in P \cap m^+(b)$ and take $\gamma_o = b + \frac{|m^+(b)|}{2}$.

Since

$$|f(\gamma+t)+f(\gamma-t)| \lesssim |f(\gamma_o+t)+f(\gamma-t)-2f(b)| + |f(\gamma_o+t)+f(\gamma-t)-2f(b)| + |f(\gamma_o+t)+f(\gamma_o-t)|$$

Then

$$\left\{\int_0^\delta |f(\gamma+t)+f(\gamma-t)|^\omega dt\right\}^{\frac{1}{\omega}}$$

$$\lesssim \left\{\int_{\gamma_o-b}^{\gamma_o-b+\delta} |f(b+u)+f(b-u)|^\omega du\right\}^{\frac{1}{\omega}}$$

$$+ \left\{\int_{\gamma-b}^{\gamma-b+\delta} |f(b+u)+f(b-u)|^\omega du\right\}^{\frac{1}{\omega}}$$

$$+ \left\{\int_0^\delta |f(\gamma_o+t)+f(\gamma_o-t)|^\omega dt\right\}^{\frac{1}{\omega}}$$

Since δ is positive infinitesimal then $\gamma_o \in A_\omega$.

Since $P \cap m^+(b)$ is uncountable, then the set $A_\omega \cap (a,b)$ is uncountable, which is contradiction with $A_\omega \cap (a,b) \subset \mathbb{N}$. Now, let $B_i = {A_i}^c = [0,1] \backslash A_i$ for all $i = 1,2,\dots,\omega$. Then on each component B_i the function f is continuous.

Applying Robinson's lemma we get that f is continuous on each component of B_i up to an unlimited index ω.

Hence f is Baire*1 on each component of B_i. Since A_ω is closed and countable it follows that f is Baire*1 on [0,1].■

Definition 2.2
A function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *infinitesimally symmetric* at $x \in \mathbb{R}$ if $f(x+\delta)+f(x-\delta)-2f(x) \cong 0$ for every infinitesimal δ and it said to be *infinitesimally smooth* at $x \in \mathbb{R}$ if

$$\frac{1}{\delta}(f(x+\delta)+f(x-\delta)-2f(x)) \cong 0.$$

Definition 2.3
Let $f: [a,b] \longrightarrow \mathbb{R}$ be an internal function. Then we say that f is $L_\omega - differentiable\ of\ first\ order$ at $x_0 \in \mathbb{R}$, and denoted by $\acute{f}_{L_\omega}(x_o)$, if there are $a_0, a_1 \in \mathbb{R}$ such that

$$\left\{\frac{1}{2\delta}\int_{-\delta}^{\delta}|f(x_0+t)-a_0-a_1t|^{\omega}dt\right\}^{\frac{1}{\omega}} \cong 0 \quad ...(1)$$

where δ is infinitesimal and ω is unlimited.

Remark 2.1:
From the classical definition of derivation and Taylor polynomial we have $a_1 = \acute{f}_{L_\omega}(x_o)$, then we can rewrite (1) as follows:

$$\left\{\frac{1}{2\delta}\int_{-\delta}^{\delta}\left|f(x_0+t)-a_0-\left(\acute{f}_{L_\omega}(x_o)\right)t\right|^{\omega}dt\right\}^{\frac{1}{\omega}} \cong 0$$

Definition 2.4
Let A be an internal real set and let $x_0 \in \mathbb{R}$. The *infinitesimal lower* and *infinitesimal upper* right density of A at x_0 are defined as follows:

$$\underline{ds_+}(A,x_o) = inf\,\frac{A \cap (x_0, x_0+\delta)}{\delta},$$

$$\overline{ds_+}(A,x_o) = sup\,\frac{A \cap (x_0, x_0+\delta)}{\delta},$$

for any infinitesimal δ.

Definition 2.5
Let $\underline{ds_+}(A,x_o)$ and $\overline{ds_+}(A,x_o)$ be such as defining in the Definition 2.4.
If $\underline{ds_+}(A,x_o) \cong \overline{ds_+}(A,x_o) \cong 1$, then x_0 is said to be an *infinitesimally density point* (or a shadow of density point) for A, denoted by $ds_+(A,x_o) \cong 1$, and is said to be an *infinitesimally dispersion point* (or a shadow of dispersion point) if $\underline{ds_+}(A,x_o) \cong \overline{ds_+}(A,x_o) \cong 0$.

Remark 2.2:
From Definition 2.5 we obtain that:

1- If x_0 is an infinitesimally density point for A, then $\underline{ds_+}(A,x_o),\ \overline{ds_+}(A,x_o) \in m(1)$.
2- If x_0 is an infinitesimally dispersion point for A, then $\underline{ds_+}(A,x_o), \overline{ds_+}(A,x_o) \in m(0)$.

Definition 2.6
Let $f: [a,b] \longrightarrow \mathbb{R}$, and $x_0 \in [a,b]$. Then f is said to be *infinitesimally approximate continuous* at x_0 if there is a set E such that $ds_+(E,x_o) \cong 1$ and $f|E$ is continuous at x_0. If f is infinitesimally approximate continuous at every point $x \in [a,b]$, then we say that f is infinitesimally approximate continuous.

Definition 2.7
Let $f: [a,b] \longrightarrow \mathbb{R}$, and $x_0 \in [a,b]$. If there exists an internal set A_{x_o} such that $ds_+\left(A_{x_o}, x_o\right) \cong 1$ and $\frac{f(x)-f(x_o)}{x-x_0} \cong \lambda$ for $x \cong x_0$. Then we call λ the *infinitesimally approximate derivative* of f at x_0, denoted by $\acute{f}_{iad}(x_o)$. If f is infinitesimally approximately drivable at each point $x_0 \in [a,b]$, then we say that f is infinitesimally approximately drivable.

Definition 2.8
Let $f: [a,b] \longrightarrow \mathbb{R}$, and $x_0 \in [a,b]$. We say that f is *infinitesimally approximately smooth* on $[a,b]$ if for each $x \in (a,b)$ the set $\{\delta: f(x+\delta)+f(x-\delta)-2f(x) \cong 0^+\}$ has only zero as a standard infinitesimally dispersion point.

Theorem 2.2
Let $f: [a,b] \longrightarrow \mathbb{R}$ be s-continuous and an infinitesimally approximately smooth function on $[a,b]$. Then $\acute{f}_{iad}(x_o)$ exists on $M(f)$ and $M(f)$ is the power of the continuum in each interval, where $M(f)$ denote the set of points $x_0 \in [a,b]$ such that for some $x \in \mathbb{R}$, $f(x)-tx$ has a local extremum at x_0.

Proof
Let $x_0 \in M$ and g be another function defined by $g(x) = f(x) - tx$, $t \in \mathbb{R}$. Then g has a local extremum, say maximum at x_o.
To obtain the result it is enough to prove that $\text{sh}\left(\acute{f}_{iad}(x_o)\right) = 0$. Let
$E = \{\delta: g(x_o+\delta)+g(x_o-\delta)-2g(x_o) \cong 0^+\}$
and $F = \{\delta: g(x_o+\delta)-g(x_o) \cong 0^+\}$.
From Definition 2.5 we obtain that zero is the standard dispersion point of E.
Now, it is remains to show that $F \cap m(0) \subset E$.
Since $g(x_o)$ is a local maximum, there is a positive infinitesimal β such that
$g(x_o \pm \delta) - g(x_o) \cong 0^-$ for each
$\delta \in micromonad\ (\beta)$. It follows that

$$\left|\frac{g(x_o+\delta)-g(x_o)}{\delta}\right| \leq \left|\frac{g(x_o+\delta)+g(x_o-\delta)-2g(x_o)}{\delta}\right| \cong 0,$$

and $\acute{f}_{iad}(x_o) \cong t$. Thus, if $0 \lesssim a < b \lesssim 1$ and $m_{ab} \cong \frac{f(b)-f(a)}{b-a}$. By mean value theorem we obtain that there is a point $x_o \in (a,b)$ such that $\acute{f}_{iad}(x_o) \cong \frac{f(b)-f(a)}{b-a}$, hence $\acute{f}_{iad}(x_o) \cong m_{ab}$. Therefore, $M(f)$ is the power of the continuum.■

Lemma 2.3
Let $f:[a,b] \longrightarrow \mathbb{R}$ be an s-continuous and $L_\omega - smooth$ function on $[a,b]$. Then the set $S = \{x \in \mathbb{R}: \acute{f}_{L_\omega}(x) \text{ is exists}\}$ is of the power of the continuum in each interval.

Proof
We will try to show that $M(f) \subset S$, where $M(f)$ is same as define in Theorem 2.2. Let $x_o \in M(f)$. Then there is $t \in \mathbb{R}$ such that $g(x) = f(x) - tx$ has a local extremum at x_o.
By Theorem 2.2 we have $\acute{f}_{iad}(x_o) \cong t$ and for positive infinitesimal ε we have

$$\begin{aligned} &\left|f(x_o+\varepsilon)-f(x_o)-\acute{f}_{iad}(x_o)\varepsilon\right| \\ &\cong |f(x_o+\varepsilon)-f(x_o)-t\varepsilon| \\ &= |g(x_o+\varepsilon)-g(x_o)| \\ &\lesssim |g(x_o+\varepsilon)+g(x_o-\varepsilon)-2g(x_o)| \\ &= |f(x_o+\varepsilon)+f(x_o-\varepsilon)-2f(x_o)|. \end{aligned}$$

Thus

$$\left|f(x_o+\varepsilon)-f(x_{o)}-\acute{f}_{iad}(x_o)\varepsilon\right| \lesssim |f(x_o+\varepsilon)+f(x_o-\varepsilon)-2f(x_o)|,$$

therefore

$$\left\{\frac{1}{2\delta}\int_{-\delta}^{\delta}\left|f(x_0+\varepsilon)-f(x_0)-\acute{f}_{iad}(x_o)\varepsilon\right|^{\omega}dt\right\}^{\frac{1}{\omega}} \cong 0,$$

for $x_o \in M(f)$. That is $M(f) \subset S$. ■

Theorem 2.4
Let $f: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be an s-continuous and $L_\omega - smooth$ function. Let $S = \{x \in \mathbb{R}: \acute{f}_{L_\omega}(x) \text{ is exists}\}$. Then $\acute{f}_{L_\omega}(x)$ has the infinitesimal intermediate value property on S.

Proof
Let $a, b \in S$, and $a < b$. Since f is s-continuous on I, then $\acute{f}_{L_\omega}(x) \cong \acute{f}_{iad}(x)$ for all $x \in S$.
Let $\lambda \in \mathbb{R}$ such that $\acute{f}_{iad}(a) \lesssim \lambda \lesssim \acute{f}_{iad}(b)$. We need to show that there exist $a \lesssim \alpha \lesssim b$ such that $\acute{f}_{iad}(\alpha) \cong \lambda$.
Without loss of generality we assume that $\lambda = 0$.
Now, assume that $f(b) \gtrsim f(a)$.
If $f(a) = f(b)$, then by Theorem 2.2 there exist a point $\alpha \in (a,b) \cap M$ such that $\acute{f}_{iad}(\alpha) = 0$. By Theorem 2.3, $M(f) \subset S$ and $\acute{f}_{L_\omega}(x) = \acute{f}_{iad}(\alpha)$.
If $f(b) > f(a)$, then from our assumption, $\acute{f}_{iad}(a) \lesssim \lambda \lesssim \acute{f}_{iad}(b)$, there is a point $c \in (a,b)$ such that $f(c) \cong f(a)$. Hence, in view of our assumption, there is a point $\alpha \in (a,c)$ such that $\acute{f}_{L_\omega}(\alpha) = \acute{f}_{iad}(\alpha) \cong 0$.
Hence $\acute{f}_{L_\omega}(\alpha)$ has the infinitesimal intermediate value property on S. ■

Throughout the following theorem, which is one of the main results of this paper, we reduce the condition of s-continuity to the infinitesimal intermediate value property condition.

Theorem 2.5
Let $f: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be $L_\omega - smooth$ function and it has infinitesimal intermediate value property on I. Let $S = \{x \in \mathbb{R}: \acute{f}_{L_\omega}(x) \text{ is exists}\}$.
Then $\acute{f}_{L_\omega}(x)$ has the infinitesimal intermediate value property on S.

Proof
Let f be an $L_\omega - smooth$ function and it has the infinitesimal intermediate value property on I. If f is s-continuous, then by Theorem 2.4 the result is follows. Now, assume that f is not s-continuous but satisfying the infinitesimal intermediate value property on I.
From Theorem 2.1 we obtain that f is Baire*1, then by [Theorem 1, (O'Malley, 1976)], f cannot be monotone. Hence it is possible to select two points η and μ with $a < \eta < \mu < b$, for $a, b \in I$ such that $f(\eta) = f(\mu)$.
Now, since f is Baire*1, and it has the infinitesimal intermediate value property on I, then by [Theorem 3, (O'Malley, 1976)], there is a point $x_o \in (\eta, \mu)$ in which f has a local maximum or local minimum. Then for unlimited ω and a positive infinitesimal ε we have

$$|f(x_0+t)-f(x_0)|^{\omega} \leq |f(x_0+t)+f(x_0-t)-2f(x_0)|^{\omega},$$

for all $t \in m(\varepsilon)$. Hence $\acute{f}_{L_\omega}(x)$ is exists. Thus f is a smooth function. Since every derivative has the intermediate value property, therefore the set $\{x \in \mathbb{R}: \acute{f}(x) \text{ is exists}\}$ has the intermediate value property and by transfer principle we obtain that f has s-derivative at each point $x \in I$. Consequentially we get that $\acute{f}_{L_\omega}(x)$ has the infinitesimal intermediate value.■

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