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Steiner Wiener Index of Certain Windmill Graphs.

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A B S T R A C T:

For a connected graph G of order p and a non-empty subset S of the vertex set of G, the n- Steiner distance of S is defined to be the smallest size among all connected sub-graphs whose vertex sets contain S. In this paper, we obtain the Hosoya polynomials of Steiner n -distance of some windmill graphs. Moreover, the Steiner Wiener indices of certain windmill graphs are also obtained.

KEY WORDS: Steiner distance, Steiner Wiener index, windmill graphs.,
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1. INTRODUCTION

In this research paper, we consider only undirected, finite, and simple graphs. We refer the reader to (Buckly, F. and Harary, F.,1990, Chartrand, G. and Lesniak, L.,1986) for unknown concepts and terminologies on graph theory. The Steiner distance in a graph, introduced in 1989 by Chartrand, Oellermann, Tian, and Zou is a generalization of the ordinary distance (Chartrand, G. Oellermann, O.R. ,Tian, S and Zou, H.B.,1989). For a connected graph G of order p and a non-empty subset of vertices $S \subseteq \mathrm{V(G)}$, the n-Steiner distance of S in G denoted by $d_G(S)$ or simply $d(S)$ is defined to be the smallest size among all connected sub-graphs $T(S)$ whose vertex sets contain S. The subgraph $T(S)$ is a tree called a Steiner tree of S. If $|S| = 2$, then the definition of the Steiner distance is the ordinary distance between the two vertices of G. For $2 \le n \le p$ and $|S| = n$, the Steiner distance of S is called the Steiner n-distance of S in G or simply the Steiner distance of S in G. The Steiner problem is the problem of finding the Steiner distance of a non-empty subset of vertices and the Steiner problem is an NP-complete problem (Gary, M. R. and Johson, D. S.,1979).

The Steiner n-eccentricity $e_n^*(v)$ of a vertex v of G denoted by $e_n^*(v)$ is defined by $e_n^*(v) = max\{d(S)|S \subseteq \mathrm{V(G)}, |S| = n \text{ and } v \in S\}$. The Steiner n-radius of G, denoted by $rad_n^*(G)$ is defined by $rad_n^*(G) = min\{e_n^*(v)|v \in \mathrm{V(G)}\}$. The Steiner n-diameter of G, denoted by $diam_n^*(G)$ or $\delta_n^*(G)$ is defined by $diam_n^*(G) = max\{e_n^*(v)|v \in \mathrm{V(G)}\}$ or it is defined to be the maximum Steiner n-distance of all n-subsets of $V(G)$, that is

$diam_n^*(G) = max\{d(S): S \subseteq \mathrm{V(G)}, |S| = \mathrm{n}\}$,

(Ali, A.A. and Saeed, W.A, 2006). The Steiner n-distance of a vertex $v \in V(G)$, denoted by $W_n^*(v, G)$ is the sum of the Steiner n-distances of all n-subsets of V(G) containing v.

The Steiner Wiener index (Herish, O.A., 2009, Xueliang, L., Yaping, M. and Ivan, G., 2016, Danklemann, P., Oellermann, O. R., and Swart, H.C.,1996) of a graph G is a graph invariant denoted by $W_n^*(G)$ and defined to be the sum of Steiner n-distances of all non-empty n-subsets of $V(G)$, that is

$W_n^*(G) = \sum_{S \subseteq \mathrm{V(G)}, |S| = \mathrm{n}} d(S)$. While the average Steiner n-distance of a graph G, denoted by $\mu_n^*(G)$, is the average of the Steiner n-distances of all n-subsets of $V(G)$, that is

$$\mu_n^*(G) = \binom{p}{n}^{-1} \sum_{S \subseteq \mathrm{V(G)}, |S| = \mathrm{n}} d(S).$$

Notice that, $W_n^*(G) = \sum_{S \subseteq \mathrm{V(G)}, |S| = \mathrm{n}} d(S)$

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$= \frac{1}{n}\sum_{v\in V(G)} W_n^*(v,G) = \binom{p}{n}\mu_n^*(G), 2 \le n \le p.$
Now, let $C_n^*(G,k)$ be the number of n-subsets of distinct vertices of G with Steiner n-distance k. The graph polynomial defined by

$$H_n^*(G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(G,k)\, x^k,$$

where δ_n^* is the Steiner n-diameter of G; is called the Hosoya polynomial of Steiner distance of G (Ali, A.A. and Saeed, W.A, 2006). Then the Steiner Wiener index of G, $W_n^*(G)$ will be $W_n^*(G) = \sum_{k=n-1}^{\delta_n^*} kC_n^*(G,k)$.
The following proposition summarizes some properties of $H_n^*(G;x)$.

Proposition **1.1.**(Ali, A.A. and Saeed, W.A, 2006) For $2 \le n \le p$, we have:
(1) $\sum_{k=n-1}^{\delta_n^*} C_n^*(G,k) = \binom{p}{n}$.
(2) $W_n^*(G) = \frac{d}{dx} H_n^*(G;x)|_{x=1}$.
(3) $H_2^*(G;x) = H(G;x) - p$, in which $H(G;x)$ is Hosoya polynomial of G with respect to the ordinary distance.■
Let $C_n^*(u,G,k)$ denote the number of n-subsets S of distinct vertices of G containing u at Steiner n-distance k.
Notice that, $C_1^*(u,G,0) = 1$.
Let us define

$$H_n^*(u,G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(u,G,k)\, x^k.$$

It is obvious that
$H_n^*(G;x) = \frac{1}{n}\sum_{u\in V(G)} H_n^*(u,G;x)$ for $n = 2,\dots,p$.

Ali and Saeed (Ali, A.A. and Saeed, W.A, 2006) were the first who studied this distance-based graph polynomial and obtained Hosoya polynomials of Steiner n-distance for some special graphs and Gutman's compound graphs.

Many authors studied Steiner distance of graphs. Dankelmann, Oellermann, and Swart introduced the concept of the average Steiner distance of graphs in (Danklemann, P., Oellermann, O. R., and Swart, H.C.,1996). In (Herish, O.A.,2009) Herish obtained Hosoya polynomials of Steiner n-distance and Steiner Wiener index of the sequential join of graphs. Yaping and Boris (Gary, M. R. and Johson, D. S., 1979) summarized the known results on the Steiner distance parameters, including Steiner distance, Steiner diameter, and Steiner Wiener index. Xueliang, Yaping, and Ivan (Xueliang, L., Yaping, M. and Ivan, G., 2016) obtained the Steiner Wiener index for some special graphs and give sharp upper and lower bounds of the Steiner Wiener index of graphs. Finally, there are a lot of recent papers related to this topic that has many applications, see (Yaping, M., Boris, F., 2021, Izudin, R., Yaping, M., Zhao W. and Boris F., 2020, Babu, A. and Baskar, J. B.,2019, Ali A. , Ravindra B. B. and Shivani G., 2022, Patrick, A. and Edy T. B, 2022).

Before closing this section, we present the following useful formulas.

Theorem **1.2.**(Ali, A.A. and Saeed, W.A,2006) Let C_p be a cycle of p vertices, then for $2 \le n \le p$, we have
(1) $H_n^*(C_p;x) = \frac{p}{n}\sum_{k=n-1}^{p-\lceil \frac{p}{n}\rceil} N(k,p:n)x^k$.
(2) $H_2^*(C_p;x)$

$$= \begin{cases} p\left(x + \cdots + x^{\frac{p-1}{2}}\right), & \text{if } p \text{ is odd}, \\ p\left(x + \cdots + x^{\frac{p}{2}-1} + \frac{1}{2}x^{\frac{p}{2}}\right), & \text{if } p \text{ is even}. \end{cases}$$

(3) $W_n^*(C_p) = \frac{p}{n}\sum_{k=n-1}^{p-\lceil \frac{p}{n}\rceil} kN(k,p:n)$.
(4) $W(C_p) = \begin{cases} \frac{p^3}{8}, & \text{if } p \text{ is odd}, \\ \frac{p(p^2-1)}{8}, & \text{if } p \text{ is even}. \end{cases}$
In which $\lceil x \rceil$ is the ceiling function that outputs the least integer greater than or equal to x, and $N(k,p:n)$ is the number of ordered partitions of p into n positive integers $(l_1, l_2, \dots, l_n)$ such that $\max l_i = p - k$.■

2. French Windmill Graphs

The French windmill graph (V. R. Kulli, Praveen Jakkannavar and B. Basavanagoud, 2019) F_t^m is the graph obtained by taking $m \ge 2$ copies of the complete graph K_t, $t \ge 2$ with a vertex say u_0 in common. The graph F_t^m is depicted in the following figure.

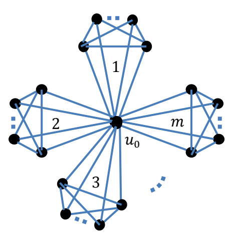


Figure 1. French windmill graph F_t^m

It is clear from Figure 1 that, for $m, t \geq 2$, the French windmill graph F_t^m has $m(t-1)+1$ vertices and $\frac{1}{2}mt(t-1)$ edges.

One can easily check that if $2 \leq n \leq m(t-1)+1$, then the Steiner n-diameter of F_t^m is $diam_n^*(F_t^m) = n$.

Let us denote V_i to be the vertex set of the i_{th} copy of K_t in F_t^m, for $i = 1,2,\dots,m$, then the Hosoya polynomial of Steiner distance of F_t^m is given in the next result.

Proposition **2.1**. For $m, t \geq 2$ and $2 \leq n \leq t$, we have

$H_n^*(F_t^m; x) = C_1 x^{n-1} + C_2 x^n$, in which

$C_1 = m\binom{t-1}{n} + \binom{m(t-1)}{n-1}$ and

$C_2 = \binom{m(t-1)}{n} - m\binom{t-1}{n}$.

Proof. Let S be any n-subset of vertices of $V(F_t^m)$, then

$$d(S) = \begin{cases} n-1, \text{if } u_0 \in S \text{ or } \{u_0 \notin S \text{ and } S \subseteq V_i\}, \\ n, \quad otherwise. \end{cases}$$

Therefore, $H_n^*(F_t^m; x) = C_1 x^{n-1} + C_2 x^n$.

Now, if $u_0 \in S$, then u_0 is adjacent to every vertex of $V(F_t^m) - \{u_0\}$ and S is connected, so $d(S) = n-1$ and this gives us $\binom{m(t-1)}{n-1}$ n-subsets S. If $u_0 \notin S$ and $S \subseteq V_i - \{u_0\}$ for $i = 1,2,\dots,m$, then S is connected, $d(S) = n-1$, and this gives us $m\binom{t-1}{n}$ such n-subsets S.

Therefore, $C_1 = m\binom{t-1}{n} + \binom{m(t-1)}{n-1}$.

Since $C_1 + C_2 = \binom{m(t-1)+1}{n}$, then the result is obtained.■

Proposition **2.2** For $m, t \geq 2$ and $t < n \leq m(t-1)$, we have

$H_n^*(F_t^m; x) = C_1 x^{n-1} + C_2 x^n$, in which

$C_1 = \binom{m(t-1)}{n-1}$ and $C_2 = \binom{m(t-1)}{n}$.

Proof. Let S be any n-subset of vertices of $V(F_t^m)$, then $d(S) = \begin{cases} n-1, & \text{if } u_0 \in S, \\ n, & \text{if } u_0 \notin S. \end{cases}$

Therefore, $H_n^*(F_t^m; x) = C_1 x^{n-1} + C_2 x^n$.

Now, if $u_0 \in S$, then S is connected, $d(S) = n-1$, and this gives us $\binom{m(t-1)}{n-1}$ such n-subsets S. Therefore, $C_1 = \binom{m(t-1)}{n-1}$.

Since $C_1 + C_2 = \binom{m(t-1)+1}{n}$, then the result is obtained.■

Corollary **2.3.** For $m, t \geq 2$ we have the Steiner Wiener index of F_t^m is given as follows

(1) If $2 \leq n \leq t$, then

$$W_n^*(F_t^m) = n\binom{m(t-1)+1}{n} - m\binom{t-1}{n} - \binom{m(t-1)}{n-1}.$$

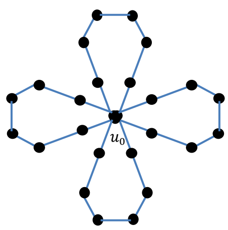
(2) If $t < n \leq m(t-1)$, then

$$W_n^*(F_t^m) = n\binom{m(t-1)+1}{n} - \binom{m(t-1)}{n-1}.$$

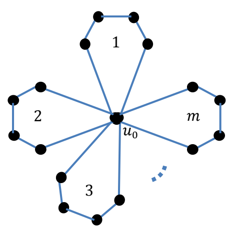
Proof. Taking the derivatives of the formulas given in Propositions 2.1 and 2.2 at $x = 1$ and simplifying, we get the results as given in the statement of the proposition.■

3. Dutch Windmill Graphs

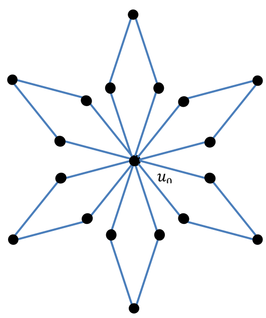
The Dutch windmill graph (V. R. Kulli, Praveen Jakkannavar and B. Basavanagoud, 2019), denoted by D_t^m is the graph obtained by taking $m \geq 2$ copies of C_t, $t \geq 3$ with a vertex say u_0 in common. The graph D_t^m for different values of m and t is depicted in the following figure.



(a) D_7^4



(b) D_t^m



(c) D_4^6

Figure 2. Dutch windmill graph

It is clear from Figure 2(b) that, for $m \geq 2$ and $t \geq 3$, the Dutch windmill graph D_t^m has $m(t-1)+1$ vertices and mt edges.
Next, we compute $diam_n^*(D_t^m)$.

***Proposition* 3.1**. If $m \geq 2$, $t \geq 3$ and $2 \leq n \leq m$ then we have,

$$diam_n^*(D_t^m) = n\left\lfloor\frac{t}{2}\right\rfloor,$$

in which $\lfloor x \rfloor$ is the floor function that outputs the greatest integer less than or equal to x.

Proof. We refer to Figure 2(b) and denote V_i to be the vertex set of the i_{th} copy of C_t in D_t^m, and let S be any n-subset of vertices of D_t^m then for $m \geq 2$, $t \geq 3$ and $2 \leq n \leq m$, the n-subset S that has maximum Steiner distance contains at most one vertex from $V_i - \{u_0\}$ say w_i. Henceforth, the Steiner tree of S must contain a spanning tree F of D_t^m; and each vertex w_i of S in $V_i - \{u_0\}$ will move through the vertices of the i_{th} copy of C_t then to connect directly through the vertex u_0.
Therefore, $d(S) = \left\lfloor\frac{t}{2}\right\rfloor + \left\lfloor\frac{t}{2}\right\rfloor + \cdots + \left\lfloor\frac{t}{2}\right\rfloor$ n-times.
This completes the proof.■

Now to find $H_n^*(D_t^m;x)$ for all $n \geq 4$, we have too many possibilities for the n-subsets S. For this reason, we shall consider only the case $n=3$ and $H_3^*(D_t^m;x)$ is given in the next proposition.

***Proposition* 3.2**. If $m \geq 2$ and $t \geq 3$, then $H_3^*(D_t^m;x)$ is given as follow

$$H_3^*(D_t^m;x) = mH_3^* + \frac{2m(m-1)}{t^2}H_2^*(C_t;x)\{H_2^*(C_t;x) + 3H_3^*(C_t;x)\} + \frac{8}{t^3}\binom{m}{3}\{H_2^*(C_t;x)\}^3.$$

Proof. Let $S = \{x_1, x_2, x_3\}$ be any 3-subset of vertices of $V(D_t^m)$, then we consider the following cases

Case1. If $S \subseteq V_i$ for $i = 1,2,\ldots,m$ then we get the polynomial $F_1(x) = mH_3^*(C_t;x)$.

Case2. If $x_1 = u_0$, $x_2 \in V_i - \{u_0\}$ and $x_3 \in V_j - \{u_0\}$ with $1 \leq i < j \leq m$ then $d(S) = d_{C_t}^*(u_0,x_2) + d_{C_t}^*(u_0,x_3)$ and the number of such 3-subsets S is $\binom{m}{2}$ and this produces the polynomial

$$F_2(x) = \binom{m}{2}H_2^*(u_0,C_t;x)H_2^*(u_0,C_t;x) = \binom{m}{2}\{H_2^*(u_0,C_t;x)\}^2.$$

Case3. If $x_1, x_2 \in V_i - \{u_0\}$ and $x_3 \in V_j - \{u_0\}$ for $1 \leq i < j \leq m$ (or $x_1 \in V_i - \{u_0\}$ and $x_2, x_3 \in V_j - \{u_0\}$) then $d(S) = d_{C_t}^*\{u_0,x_1,x_2\}$+ $d_{C_t}^*(u_0,x_3)$, the number of such 3-subsets S is $2\binom{m}{2}$ and this produces the following polynomial

$$F_3(x) = 2\binom{m}{2}H_3^*(u_0,C_t;x)H_2^*(u_0,C_t;x).$$

Case4. If $x_1 \in V_i - \{u_0\}$, $x_2 \in V_j - \{u_0\}$ and $x_3 \in V_k - \{u_0\}$ for $1 \leq i < j < k \leq m$ then $d(S) = d_{C_t}^*(u_0,x_1) + d_{C_t}^*(u_0,x_2) + d_{C_t}^*(u_0,x_3)$, the number of such 3-subsets S is $\binom{m}{3}$ and this produces the following polynomial

$$F_4(x) = \binom{m}{3}H_2^*(u_0,C_t;x)H_2^*(u_0,C_t;x)H_2^*(u_0,C_t;x) = \binom{m}{3}\{H_2^*(u_0,C_t;x)\}^3.$$

Now, combining the above four cases we get

$$H_3^*(D_t^m;x) = mH_3^*(C_t;x) + \binom{m}{2}\{H_2^*(u_0,C_t;x)\}^2 + 2\binom{m}{2}H_3^*(u_0,C_t;x)H_2^*(u_0,C_t;x) + \binom{m}{3}\{H_2^*(u_0,C_t;x)\}^3.$$

Substituting $H_3^*(C_t;x) = \frac{t}{3}H_3^*(u_0,C_t;x)$ and $H_2^*(C_t;x) = \frac{t}{2}H_2^*(u_0,C_t;x)$ in the above formula and then simplifying we get the result as given in the statement of the proposition.■

Next, we compute 3-Steiner Wiener index of D_t^m.

***Corollary* 3.3.** For $m,t \geq 2$ we have the 3-Steiner Wiener index of the Dutch windmill graph is given by

$$W_3^*(D_t^m) = \frac{m}{t^2}W_3^*(C_t)\{t^2 + 3t(t-1)(m-1)\} + \frac{(t-1)^2 m!}{t(m-3)!}W_2^*(C_t) + \frac{2}{t^2}m(m-1)W_2^*(C_t)\left\{t(t-1) + 3\binom{t}{3}\right\}.$$

Proof. Taking the derivative of $H_3^*(D_t^m;x)$ given in Propositions 3.2 at $x=1$, we get

$$W_3^*(D_t^m) = mW_3^*(C_t) + \frac{2}{t^2}m(m-1)W_2^*(C_t)\{H_2^*(C_t;1) + 3H_3^*(C_t;1)\} + \frac{6}{t^2}m(m-1)H_2^*(C_t;1)W_3^*(C_t) + \frac{24}{t^3}\binom{m}{3}H_2^*(C_t;1)\}^2W_2^*(C_t).$$

Now, substituting $H_2^*(C_t;1) = \binom{t}{2}$, $H_3^*(C_t;1) = \binom{t}{3}$ and simplifying, we get the result as given in the statement of the corollary.■

4. Kulli-Cycle Windmill Graphs

The Kulli-cycle windmill graph (V. R. Kulli, Praveen Jakkannavar and B. Basavanagoud, 2019) C_{t+1}^{m} is the graph obtained by taking m copies of $C_t + K_1$, $t \geq 3$ with a vertex u_0 of K_1 in common. The graph C_{t+1}^{m} is depicted in Figure 3.
It is clear from Figure 3 that, for $m \geq 2$ and $t \geq 3$, the Kulli-cycle windmill graph C_{t+1}^{m} has $mt + 1$ vertices and $2mt$ edges.

We can easily check that, if $2 \leq n \leq mt$, then the Steiner n-diameter of C_{t+1}^{m} is $diam_n^*(C_{t+1}^{m}) = n$.

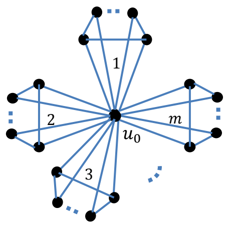


Figure 3. Kulli-cycle Windmill graph C_{t+1}^{m}

Let us denote V_i to be the vertex set of the i_{th} copy of C_t in C_{t+1}^{m}, for $i = 1,2,\dots,m$, then the Hosoya polynomial of Steiner distance of C_{t+1}^{m} is given in the next result.

Proposition **4.1**. For $m \geq 2$, $t \geq 3$, and $2 \leq n \leq t$, we have

$H_n^*(C_{t+1}^{m}; x) = C_1 x^{n-1} + C_2 x^n$, in which

$C_1 = mt + \binom{mt}{n-1}$ and $C_2 = \binom{mt}{n} - mt$.

Proof. Let S be any n-subset of vertices of $V(C_{t+1}^{m})$, then

$d(S) = n - 1$, if $u_0 \in S$ or $\{u_0 \notin S$ and S is connected subset of $V_i\}$, and

$d(S) = n$, otherwise.

Therefore, $H_n^*(C_{t+1}^{m}; x) = C_1 x^{n-1} + C_2 x^n$.

Now, if $u_0 \in S$, then u_0 is adjacent to every vertex of $V(C_{t+1}^{m}) - \{u_0\}$ and S is connected, so $d(S) = n - 1$ and this gives us $\binom{mt}{n-1}$ n-subsets S. If $u_0 \notin S$ and S is a connected subset of V_i for $i = 1,2,\dots,m$, $d(S) = n - 1$, and this gives us mt such n-subsets S.

Therefore, $C_1 = \binom{mt}{n-1} + mt$.

Since $C_1 + C_2 = \binom{mt+1}{n}$, then the result is obtained.■

Proposition **4.2**. For $m \geq 2$, $t \geq 3$ and $t < n \leq mt$, we have

$H_n^*(C_{t+1}^{m}; x) = C_1 x^{n-1} + C_2 x^n$, in which

$C_1 = \binom{mt}{n-1}$ and $C_2 = \binom{mt}{n}$.

Proof. Let S be any n-subset of vertices of $V(C_{t+1}^{m})$, then $d(S) = \begin{cases} n-1, & \text{if } u_0 \in S, \\ n, & \text{if } u_0 \notin S. \end{cases}$

Therefore, $H_n^*(C_{t+1}^{m}; x) = C_1 x^{n-1} + C_2 x^n$.

Now, if $u_0 \in S$, then S is connected, $d(S) = n - 1$, and this gives us $\binom{mt}{n-1}$ such n-subsets S and $C_1 = \binom{mt}{n-1}$.

This completes the proof.■

Corollary **4.3.** For $m \geq 2$ and $t \geq 3$ we have the Steiner Wiener index of C_{t+1}^{m} is given as follows

(1) If $2 \leq n \leq t$, then

$$W_n^*(C_{t+1}^{m}) = n\binom{mt+1}{n} - \binom{mt}{n-1} - mt.$$

(2) If $t < n \leq mt$, then

$$W_n^*(C_{t+1}^{m}) = n\binom{mt+1}{n} - \binom{mt}{n-1}.$$

Proof. Obvious.■

5. Kulli-Wheel Windmill Graphs

Let W_t be a wheel graph of order t. The kulli-wheel windmill graph W_{t+1}^{m} is the graph obtained by taking m copies of $W_t + K_1$, $t \geq 4$ with a vertex u_0 of K_1 in common. The graph W_{t+1}^{m} is depicted in Figure 4.
It is clear from Figure 4 that, for $m \geq 2$ and $t \geq 4$, the kulli-wheel windmill graph W_{t+1}^{m} has $mt + 1$ vertices and $m(3t - 2)$ edges.
We can easily check that for $2 \leq n \leq mt$, the Steiner n-diameter of W_{t+1}^{m} is $diam_n^*(W_{t+1}^{m}) = n$.

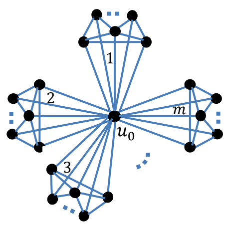


Figure 4. Kulli-wheel windmill graph W_{t+1}^{m}

Hosoya polynomial of Steiner distance of W_{t+1}^{m} is given in the next result.

Proposition **5.1**. For $m \geq 2$, $t \geq 4$, and $2 \leq n \leq t$, we have

$H_n^*(W_{t+1}^{m}; x) = C_1 x^{n-1} + C_2 x^n$, in which

$C_1 = \binom{mt}{n-1} + m\binom{t-1}{n-1} + m(t-1)$ and

$C_2 = \binom{mt}{n} - m\binom{t-1}{n-1} - m(t-1)$.

Proof. We refer to Figure 4, and for $i = 1,2,\dots,m$ denote V_i be the vertex set of the i_{th} copy of W_t in W_{t+1}^m , v_i be the center of W_t in W_{t+1}^m, and let S be any n-subset of vertices of $V(W_{t+1}^m)$, then $d(S) = n-1$, if $u_0 \in S$ or $\{u_0 \notin S$ and S is connected$\}$, and $d(S) = n$, otherwise.
Therefore, $H_n^*(W_{t+1}^m;x) = C_1x^{n-1} + C_2x^n$.
Now, if $u_0 \in S$, then u_0 is adjacent to every vertex of $V(W_{t+1}^m) - \{u_0\}$ and S is connected, so $d(S) = n-1$ and this gives us $\binom{mt}{n-1}$ such n-subsets S.
If $u_0 \notin S$ and S is connected then, $S \subseteq V_i$ and we consider two subcases as follows:
(i) If $v_i \in S$, then v_i is adjacent to every vertex of the cycle vertices of W_t, and S is connected, so $d(S) = n-1$ and this gives us $m\binom{t-1}{n-1}$ such n-subsets S.
(ii) If $v_i \notin S$, then the vertices of S are the connected vertices of the cycle vertices of W_t and this gives us $m(t-1)$ such n-subsets S.
Therefore,
$$C_1 = \binom{mt}{n-1} + m\binom{t-1}{n-1} + m(t-1).$$

Since $C_1 + C_2 = \binom{mt+1}{n}$, then the result is obtained.■

Proposition **5.2**. For $m \geq 2$, $t \geq 4$ and $t < n \leq mt$,
$H_n^*(W_{t+1}^m;x) = C_1x^{n-1} + C_2x^n$, in which
$C_1 = \binom{mt}{n-1}$ and $C_2 = \binom{mt}{n}$.
Proof. Let S be any n-subset of vertices of $V(W_{t+1}^m)$, then $d(S) = \begin{cases} n-1, & \text{if } u_0 \in S, \\ n, & \text{if } u_0 \notin S. \end{cases}$
Therefore, $H_n^*(W_{t+1}^m;x) = C_1x^{n-1} + C_2x^n$.
Now, if $u_0 \in S$, then the number of such n-subsets S is $\binom{mt}{n-1}$ and $C_1 = \binom{mt}{n-1}$.
This completes the proof.■
Corollary **5.3.** For $m, t \geq 2$ we have the Steiner Wiener index of the graph W_{t+1}^m is given as follows
(1) If $2 \leq n \leq t$, then
$$W_n^*(W_{t+1}^m) = n\binom{mt+1}{n} - \binom{mt}{n-1} - m\binom{t-1}{n-1} - m(t-1).$$
(2) If $t < n \leq mt$, then
$$W_n^*(W_{t+1}^m) = n\binom{mt+1}{n} - \binom{mt}{n-1}.$$
Proof. Obvious.■

6. Kulli-Path Windmill Graphs

The Kulli-path windmill graph (V. R. Kulli, Praveen Jakkannavar and B. Basavanagoud, 2019) P_t^m is the graph obtained by taking m copies of $P_t + K_1$, $t \geq 2$ with a vertex u_0 of K_1 in common. The graph P_{t+1}^m is depicted in Figure 5.
It is clear from Figure 5 that, for $m, t \geq 2$, the kulli-path windmill graph P_{t+1}^m has $mt+1$ vertices and $m(2t-1)$ edges.We can easily check that if $2 \leq n \leq mt$, then $diam_n^*(P_{t+1}^m) = n$.

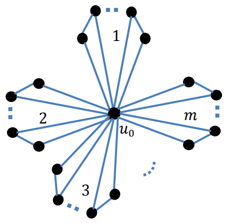


Figure 5. Kulli-Path Windmill Graph P_{t+1}^m

Let us denote V_i to be the vertex set of the i_{th} copy of P_t in P_{t+1}^m, for $i = 1,2,\dots,m$, then the Hosoya polynomial of Steiner distance of P_{t+1}^m is given in the next result.
Proposition **6.1**. For $m \geq 2$, $t \geq 3$ and $2 \leq n \leq t$, we have
$H_n^*(P_{t+1}^m;x) = C_1x^{n-1} + C_2x^n$, in which
$C_1 = \binom{mt}{n-1} + m(t-n+1)$ and
$C_2 = \binom{mt}{n} - m(t-n+1)$.

Proof. Let S be any n-subset of vertices of $V(P_{t+1}^m)$, then $d(S) = n-1$, if $u_0 \in S$ or $\{u_0 \notin S$ and S is connected$\}$, and $d(S) = n$, otherwise.
Therefore, $H_n^*(P_{t+1}^m;x) = C_1x^{n-1} + C_2x^n$.
Now, if $u_0 \in S$, then the number of such n-subsets S is $\binom{mt}{n-1}$ and if $u_0 \notin S$ and S is connected then the number of such n-subsets S equals $m(t-n+1)$.
Therefore, $C_1 = \binom{mt}{n-1} + t - n + 1$.
Since $C_1 + C_2 = \binom{mt+1}{n}$, then the result is obtained.■
Proposition **6.2**. For $m \geq 2$, $t \geq 3$ and $t < n \leq mt$, we have
$H_n^*(P_{t+1}^m;x) = C_1x^{n-1} + C_2x^n$, in which
$C_1 = \binom{mt}{n-1}$ and $C_2 = \binom{mt}{n}$.

Proof. Let S be any n-subset of vertices of $V(P_{t+1}^m)$, then $d(S) = \begin{cases} n-1, & \text{if } u_0 \in S, \\ n, & \text{if } u_0 \notin S. \end{cases}$
Therefore, $H_n^*(P_{t+1}^m; x) = C_1 x^{n-1} + C_2 x^n$.
Now, if $u_0 \in S$, then the number of such n-subsets S is $\binom{mt}{n-1}$ and then $C_1 = \binom{mt}{n-1}$.
This completes the proof.■

Corollary **6.3.** For $m, t \geq 2$ we have the Steiner Wiener index of P_{t+1}^m is given as follows
(1) If $2 \leq n \leq t$, then

$$W_n^*(P_{t+1}^m) = n\binom{mt+1}{n} - \binom{mt}{n-1} - m(t-n+1).$$

(2) If $t < n \leq mt$, then

$$W_n^*(P_{t+1}^m) = n\binom{mt+1}{n} - \binom{mt}{n-1}.$$

Proof. Obvious.■

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