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Maximal Ideal Graph of Zn

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ABSTRACT

In this paper, we present the structure of the maximal ideal graph of the ring Z_n. Several graph properties are given including completeness, planarity, separability. Furthermore, we show that $MG(\mathrm{Z}_n)$ is Hamiltonian, and Eulerian under certain conditions. We also compute the degree of vertices, size, circumference, girth, and clique number of $MG(\mathrm{Z}_n)$ for specific types of n. Finally, we find Hosoya polynomial of $MG(\mathrm{Z}_n)$ which depends on both degree of vertices and size of the graph for certain values of n.

1.Introduction

Algebraic graph theory has benefited from investigation of the conclusion between rings and graph theory. The concept of graph associated to rings was first introduced by (Beck, 1988). Furthermore, the interplay between graphs and algebraic structures was investigated by (Ye and Wu, 2012), (Anderson and Badawi, 2017), (Barman and Rajkhowa, 2019), (Nadir et al., 2019), (Abdulqadr, 2020a), (Abdulqadr, 2020b), (Habibi et al., 2021), (Jorf and Oukhtite, 2024).

In (Ye and Wu, 2012), Ye and Wu introduced the co-maximal ideal graph of a commutative ring R as an undirected simple graph whose vertex set consists of all proper ideals of R not contained in the Jacobson radical of R, and two distinct vertices are adjacent if their sum equals R. The non-comaximal graph of ideals of a ring R, as defined by (Barman and Rajkhowa, 2019), is the undirected graph whose vertex set consists of all non-trivial ideals of R, when two distinct vertices are adjacent if their sum is not equal to R. In (Abdulqadr, 2020b), Abdulqadr investigated a subgraph of the non-comaximal graph of ideals of a commutative ring R, defined using the maximal ideals of R and called the maximal ideal graph of R. This graph inherits several properties from the non-comaximal graph such as connectivity, diameter, and girth. It possesses several graph theoretic properties that are established in (Abdulqadr, 2020b). The maximal ideal graph, denoted by $MG(R)$, is defined as the undirected simple graph whose vertex set consists of all non-trivial ideals of R, and two distinct vertices are adjacent if their sum is a maximal ideal of R.

In this paper, we explore some properties and characterizations of the maximal ideal graph of the ring Z_n in certain values of n.

Throughout this paper, the number n is assumed to be neither prime nor the product of two primes, as the corresponding graphs in these cases are, respectively, an empty graph and a null graph.

We begin by recalling some basic definitions in (Gary and Linda, 1986) that will be used throughout this paper as follows:

A path in a graph G is a sequence of vertices $v_1, v_2, \dots, v_n$ such that each consecutive pair (v_i, v_{i+1}) is connected by an edge, and all vertices in the sequence are distinct. A vertex u is said to be connected to a vertex v in a graph G if there exists a $u-v$ path in G. A graph G is connected if every two of its vertices are connected. The degree of a vertex v in a graph G is the number of vertices of G adjacent to v. An isolated vertex in a graph G is a vertex that has no edges connected to it. For a connected graph G, the distance between two vertices u and v denoted by $d(u,v)$ is the minimum of the lengths of the $u-v$ paths of G. The eccentricity of a vertex in a connected graph G is $e(v) = Max_{u\in V(G)} d(u,v)$, where $V(G)$ is the vertex set of G. The diameter of a connected graph G is $diam(G) = Max_{v\in V(G)} e(v)$. The girth of a graph G, denoted by $g(G)$, is the length of the shortest cycle in G. The circumference of a connected graph G, denoted by $c(G)$, is the length of the longest cycle in G. A component of a connected graph G is the maximal connected subgraph of G, and the number of components of G is denoted by $k(G)$. A vertex v in a connected graph G is called cut-vertex if $k(G-v) > k(G)$. The complement of a graph G, denoted by $\overline{G}$, is that graph of vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A graph G is n-partite, $\mathrm{n} \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, \dots, V_n$ such that every edge of G joins a vertex of V_i and a vertex of V_j, where $i \neq j$. A circuit in a connected graph G that contains every edge of G is Eulerian circuit. A graph possessing an Eulerian circuit is called Eulerian graph. A graph G is defined to be Hamiltonian if it has a cycle containing all the vertices of G. A subdivision of a graph G is a new graph created by replacing edges. Specifically, an edge is subdivided by adding a new vertex w and replacing the original edge with two new edges $\{u,v\}\text{and}\,\{w,v\}$. A graph G is planar if it can be embedded in the plane. The clique number of a graph G, denoted by $\omega(G)$, is the order of the maximal complete subgraph of G. An independent set in a graph G is a subset of vertex set of G such that no two are adjacent. A graph G is split if its vertex set can be partitioned into two disjoint sets K and I such that K is a clique and I is an independent set in G.

Hosoya polynomial is defined by

$$H(G,x) = \sum_{i=0}^{diam(G)} d(G,k)x^k,$$

where $d(G,k)$ is the number of pairs of vertices of a graph G at distance k. (See (Hosoya, 1988))

The following results are also used throughout this paper (See (Gary and Linda, 1986)):

Theorem1.1
If G is a graph of size q, then

$$\sum_{v\in V(G)} deg(v) = 2q.$$

Theorem1.2
A graph G of order $p \geq 3$ is block if and only if every two vertices of G lie on a common cycle.

Theorem1.3
A graph G is planar if G contains no subgraph that is subdivision of K_5 or $K_{3,3}$.

Theorem1.4
A non-trivial connected graph G is Eulerian if and only if every vertex of G has even degree.

Lemma1.5 (Hosoya, 1988)
Let G be a connected graph of order r. Then $\sum_{k=0}^{diam(G)} d(G,k) = \frac{1}{2}r(r+1)$.

2. Some Properties of $MG(Z_n)$

In this section, we explore some graph properties and characterizations of $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$, where $p_1, p_2, \dots, p_m$ are distinct primes and $n_1, n_2, \dots, n_m$ are positive integers.

Remark2.1
The graph $MG(Z_n)$ is a star if and only if $n = p^k$, where p is a prime number and k is an integer greater than one.

Proposition2.2
The graph $MG(Z_n)$ is complete if and only if either $n = p^2$ or $n = p^3$, where p is a prime.

Proof
First, suppose that $n = p^k$, where p is a prime and k is an integer greater than one. If $k = 2$ or $k = 3$, then the corresponding graph consists of a single vertex (p) or an edge $\{ (p), (p^2) \}$, respectively. In both cases, the graph $MG(Z_n)$ is a complete graph. If $k > 3$, then by Remark2.1, the graph $MG(Z_n)$ is a star graph of order greater than 2.
Now, suppose that n has at least two distinct prime factors, say p_1 and p_2, then $(p_1) + (p_2)$ is not a maximal ideal of Z_n. In both of these later cases, the graph $MG(Z_n)$ is not complete.

Theorem2.3
The graph $MG(Z_n)$ is block if and only if $n \notin \{p_1^2 p_2, p_1^k\}$, where p_1 and p_2 are distinct prime numbers and $k \geq 3$ is an integer.

Proof
If $n = p_1^k$ with $k \geq 3$, then $MG(Z_n)$ is a star graph. If $n = p_1^2 p_2$, then the only vertex adjacent to (p_2) is $(p_1 p_2)$. Then $(p_1 p_2)$ is a cut-vertex of $MG(Z_{p_1^2 p_2})$. In both cases, $MG(Z_n)$ is not a block.
Now, let $n \notin \{p_1^2 p_2, p_1^k\}$ with $k \geq 3$. Without loss of generality, let $n = p_1^{n_1} p_2^{n_2}$ with $n_1, n_2 \geq 2$.
Then

$C_1: (p_1), (p_1 p_2^j), (p_1^i), (p_1 p_2^l), (p_1^f), (p_1 p_2), (p_2^j), (p_1^i p_2)$ $(p_2^l$
$(p_2), (p_1^s p_2^t), (p_1)$

is a cycle in $MG(Z_{p_1^{n_1} p_2^{n_2}})$, for every $i, f = 2, 3, \dots, n_1$ and $j,\ l = 2, 3, \dots, n_2$ with $i \neq f$ and $j \neq l$.
Also,

$$C_2: (p_1), (p_1^s p_2^t), (p_2), (p_1^k p_2^m), (p_1)$$

is a cycle in $MG(Z_{p_1^{n_1} p_2^{n_2}})$, for every $s, k = 2, 3, \dots, n_1 - 1$ and $t, m = 2, 3, \dots, n_2$.
Thus, every two distinct vertices in $MG(Z_{p_1^{n_1} p_2^{n_2}})$ lie on a common cycle. By the first part of Theorem1.2, the graph $MG(Z_{p_1^{n_1} p_2^{n_2}})$ is a block.

Theorem2.4
Let $m \geq 3$ and $n_1 \geq n_2, n_3, \dots, n_m$. Then the clique number of $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$ is

$$\omega\left(MG\left(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}}\right)\right) = \begin{cases} m & if \quad n_1 = 1 \\ m+1 & if\ n_1 \geq 2 \end{cases}.$$

Proof
If $n_1 = 1$, then $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$ contains a maximal complete subgraph with vertices (p_1) and $(p_1 p_2)$, $(p_1 p_3)$, $\dots$, $(p_1 p_m)$. In this case, $\omega(MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})) = m$.
If $n_1 \geq 2$, then the graph induced by the vertex set $\{(p_1 p_2), (p_1 p_3), \dots, (p_1 p_m)\} \cup \{(p_1), (p_1^2)\}$ is the maximal complete subgraph of $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$. Thus, $\omega(MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})) = m + 1$.

Proposition2.5

The graph $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$ contains a cycle if and only if either $m > 2$, or $m = 2$ and at least one of n_1 or n_2 is greater than one. Furthermore, if $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$ contains a cycle, then the girth is equal to three.

Proof

If $m > 2$, then $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$ contains a cycle formed by the vertices (p_1), (p_1p_2), and (p_1p_3).

If $m = 2$ and n_1 >1, then $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$ contains a cycle formed by the vertices (p_1), (p_1^2), and (p_1p_2). In both cases, the girth is equal to 3.

Conversely, if $MG(Z_{p_1^{n_1}p_2^{n_2}\dots p_m^{n_m}})$ contains a cycle, then Remark2.1 implies that $m \geq 2$.

If $m = 2$, then at least one of n_1 or n_2 must be greater than one.

Theorem2.6

Let $(m, n_1) \notin \{(1,1), (1,2)\}$.

Then $MG(Z_{p_1^{n_1}p_2\dots p_m})$ is a split graph if and only if either $m = n_1 = 2$ or $(m = 1 \; and \; n_1 \geq 3)$.

Proof

If $m = 1$ and $n_1 \geq 3$, then by Remark2.1 the graph $\mathrm{MG}(\mathrm{Z}_{\mathrm{p}_1^{\mathrm{n}_1}})$ is a split graph.

Now, assume that $m = n_1 = 2$. Then the graph $\mathrm{MG}\,(\mathrm{Z}_{\mathrm{p}_1^2\mathrm{p}_2})$ consists of an end vertex (p_2) and a complete graph with vertices (p_1), (p_1p_2), and (p_1^2). In this case, $MG(Z_{p_1^2p_2})$ is a split graph.

Next, assume $m = 2 < n_1$. Then the graph contains a complete subgraph K_3 formed by vertices (p_1), (p_1^2) and (p_1p_2). Since (p_2) and $(p_1^2p_2)$ are adjacent in $MG(z_{p_1^{\mathrm{n}_1}p_2})$, the set of all vertices of $MG(z_{p_1^{\mathrm{n}_1}p_2})$ except (p_1), (p_1^2) and (p_1p_2) does not form an independent set.

Now, suppose that $m \geq 3$ and $n \geq 1$. Then the graph $MG(Z_{p_1^{n_1}p_2\dots p_m})$ contains a complete subgraph of vertex set

$$\{(p_1), (p_1^2)\} \cup \{(p_1p_2), (p_1p_3), \dots, (p_1p_m)\}.$$

Then $V(MG(Z_{p_1^{n_1}p_2\dots p_m})) - \{(p_1), (p_1^2)\} \cup \{(p_1p_2), (p_1), (p_1p_m)\}$ is not an independent set, because $(p_1p_2 \dots p_m)$ is adjacent to each of (p_2), (p_3), …, (p_m) in $MG(Z_{p_1^{n_1}p_2\dots p_m})$.

In both cases, the graph $MG(Z_{p_1^{n_1}p_2\dots p_m})$ is not a split graph.

We close this section with finding Hosoya polynomial of the graph $MG(Z_{p_1^n})$.

Theorem2.7

Hosoya polynomial of the graph $MG(Z_{p_1^n})$ is $H\big(MG\big(Z_{p_1^n}\big), x\big) = (0.5n^2 - 2.5n + 2.5\,)\mathrm{x}^2 + (n-2)x + n - 1$

Proof

First, we have the polynomial

$$H(MG(Z_{p_1^n})\,, x) = \sum_{k=0}^{2} d(MG(Z_{p_1^n})\;, k)x^k$$
$$= a_o + a_1x + a_2x^2.$$

From Remark2.1, t order and size of $MG(Z_{p_1^n})$ are

$$a_o = d\big(MG(Z_{p_1^n}), 0\big) = n - 1, \text{ and}$$
$$a_1 = d\big(MG(Z_{p_1^n}), 1\big) = n - 2, \text{ respectively.}$$

On the other hand, Lemma1.5 gives that

$$a_2 = \frac{1}{2}a_o(a_o + 1) - a_o - a_1$$
$$= 0.5n(n-1) - (n-1) - (n-2)$$
$$= 0.5n^2 - 2.5n + 2.5.$$

Thus,

$$H\big(MG\big(Z_{p_1^n}\big), x\big) = \Big(3 - \frac{n}{2}\Big)\mathrm{x}^2 + (n-2)x + n - 1$$

3. Maximal Ideal Graph of $Z_{p_1^s p_2^t}$

In this section, we present some results on the graph $MG(Z_{p_1^s p_2^t})$, where p_1 and p_2 are distinct primes, and $s, t \in Z^+$ with $(s,t) \neq (1,1)$.

Theorem3.1

In the graph $MG(Z_{p_1^s p_2^t})$,

$$deg(p_1^i p_2^j) = \begin{cases} st + s - 2 & if \; (i,j) = (1,0) \\ st + t - 2 & if \; (i,j) = (0,1) \\ s + t & if \; (i,j) = (1,1) \\ 1 + s & if \; (i = 1\,, j = 2, 3, \dots, t) \\ 2 & if \; i = 2, 3, \dots, s, j = 2, 3, \dots, t \\ & and \; (i,j) \neq (s,t) \end{cases}$$

Proof

Let $(i,j) = (1,0)$. The vertex (p_1) is adjacent to all vertices of $MG(Z_{p_1^s p_2^t})$ except for the vertices (p_2^k) where 1≤ $k \leq t$. Thus,

$$deg(p_1) = \Big|MG(Z_{p_1^s p_2^t})\Big| - 1 - t = (st + s + t - 1) - t - 1 = st + s - 2.$$

Also, for every $i = 2, 3, \dots, s$ and $j = 2, 3, \dots, t$ with $(i,j) \neq (s,t)$, the vertex $(p_1^i p_2^j)$ is adjacent only to (p_1) and (p_2) in $MG(Z_{p_1^s p_2^t})$. This means that $deg(p_1^i p_2^j) = 2$.

Similarly, the degree of each remaining vertex in

$MG(Z_{p_1^s p_2^t})$ can be determined in the same manner.

Corollary3.2

The size of $MG(Z_{p_1^s p_2^t})$ is

$$\left|E(Z_{p_1^s p_2^t})\right| = 4st - 4.$$

Proof

From Theorem1.1 and Theorem 3.1, the size of $MG(Z_{p_1^s p_2^t})$ is

$$\left|E(Z_{p_1^s p_2^t})\right| = \frac{1}{2}\sum_{v\in V(MG(Z_{p_1^s p_2^t}))} deg(v)$$

$$= \frac{1}{2}(st + s - 2 + st + t - 2 + s + t + 2(1+s)(t-1) + 2(1+t)(s-1) + 2(st - s - t))$$

$$= 4st - 4.$$

Theorem3.3

Let $s > t \geq 1$. The graph $MG(Z_{p_1^s p_2^t})$ is 3- partite.

Proof

First, we partition the vertex set of $MG(Z_{p_1^s p_2^t})$ into three subsets

$A_1 = \{(p_1), (p_2)\}$,

$A_2 = \{(p_1^2), (p_1^3), \ldots, (p_1^s)\} \cup \{(p_2^2), (p_2^3), \ldots, (p_2^t)\}$, and

$A_3 = \{(p_1^x p_2^y); x = 1, 2, \ldots, s,\ y = 1, 2, \ldots, t; (x, y) \neq (s, t)\}$.

No two distinct vertices within the same set A_r are adjacent, for every $r = 1, 2, 3$.

Thus, $MG(Z_{p_1^s p_2^t})$ is a 3-partite graph.

Proposition3.4

In the graph $MG(Z_{p_1^s p_2^t})$,

$$c(MG(Z_{p_1^s p_2^t})) = \begin{cases} 4s & if \quad s = t \\ 4s + 1 & if \quad s < t \end{cases}.$$

Proof

We give two cases for s and t:

Case1. $s = t$

Define three internally vertex-disjoint paths in $MG(Z_{p_1^s p_2^t})$:

$P: (p_1), (p_1p_2^2), (p_1^2), (p_1p_2^3), \ldots, (p_1^{s-1}), (p_1p_2^s)$,

$Q: (p_2^s), (p_1^s p_2),\ (p_2^{s-1}), (p_1^{s-1}p_2), \ldots, (p_2^2), (p_1^2 p_2)$, and

$S: (p_2), (p_1^2 p_2^2), (p_1)$

Since the endpoints of these paths are pairwise adjacent, they form a single cycle

$$C = \cup \{(p_1p_2^s),\ (p_1p_2)\} \cup Q \cup S$$

of length $(2s-3) + (2s-3) + 6 = 4s$. This cycle includes all vertices of the forms form $(p_1^i), (p_2^i), (p_1^i p_2)$, or $(p_1 p_2^i)$ for $i = 1, 2, \ldots, s$ with exactly one vertex $\left(p_1^i p_2^j\right)$ for $i, j = 1, 2, \ldots, s$ (other than (s, s)) , and that vertex is adjacent only to (p_1) and (p_2). Therefore, the circumference of the graph is $c\left(MG\left(Z_{p_1^s p_2^t}\right)\right) = 4s$.

Case2. $s <$ t

Again, construct three internally vertex-disjoint paths:

$A_1: (p_1),\ (p_1p_2^s),\ (p_1^2), (p_1p_2^{s-1}), \ldots, (p_1^s), (p_1p_2)$,

$A_2: (p_2^S), (p_1^s p_2), (p_2^{s-1}), (p_1^{s-1}p_2), \ldots, (p_2^2), (p_1^2 p_2)$, and

$A_3: (p_2^{s+1}), (p_2), (p_1^2 p_2^2), (p_1)$.

They form a cycle of length

(2s−1)+(2s−3)+3=4s+1,

covering all vertices of the forms $(p_1^i), (p_2^i), (p_1^i p_2)$, and $(p_1 p_2^i)$ for ; $i = 1, 2, \ldots, s$, plus the extra vertex (p_2^{s+1}). Moreover, the vertex $(p_1 p_2^t)$ is adjacent to both each (p_1^i)(for i=1, 2, …, s) and to (p_2) for every $i = 1, 2, \ldots, s$.

Hence, the circumference in this case

$$c\left(MG\left(Z_{p_1^s p_2^t}\right)\right) = 4s + 1.$$

Theorem3.5

Assume that $s \geq t$. Then the graph $MG(Z_{p_1^s p_2^t})$ and its complement $\overline{MG(Z_{p_1^s p_2^t})}$ are planar if and only if either $s > 1 = t$ or $s = t = 2$.

Proof

If $s = t = 2$, the graph $MG(Z_{p_1^s p_2^t})$ is again planar.

Let $s > 1 = t$. Then the following figure illustrates that $MG(Z_{p_1^s p_2})$ is a planar graph:

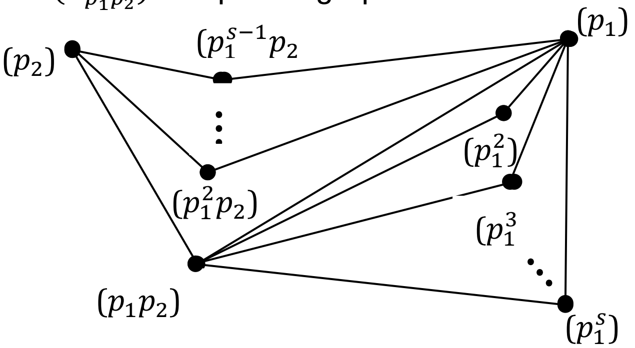

Figure1: The graph $MG(Z_{p_1^s p_2})$

If $2 < t \leq s$, then $MG(Z_{p_1^s p_2^t})$ contains a sub graph isomorphic to $K_{3,3}$ with bipartition

$S_1 = \{(p_1), (p_2^2), (p_2^3)\}$ and

$S_2=\{(p_1p_2),(p_1^2p_2),(p_1^3p_2)\}$.

In the special case $t=2<s$, the corresponding partite sets are

$S_1=\{(p_1),(p_2),(p_2^2)\}$ and
$S_2=\{(p_1p_2),(p_1^2p_2),(p_1^3p_2)\}$.

By Theorem1.3, any graph containing $K_{3,3}$ as a subgraph is nonplanar. Hence $MG\left(Z_{p_1^sp_2^t}\right)$ is non-planar.

It remains to address planarity of the complement $\overline{MG(Z_{p_1^sp_2^t})}$.

When $s=t=2$, Figure2 shows that the complement graph is planar.

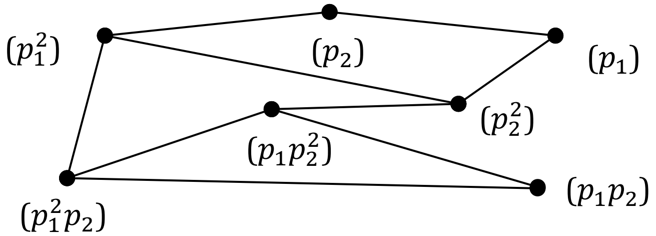

Figure2: The graph $\overline{MG(Z_{p_1^sp_2^t})}$

If $1<t<s$, then the complement contains a subgraph isomorphic to K_5 formed by vertices (p_1^2), (p_1^3), $(p_1^2p_2)$, $(p_1^3p_2)$, and $(p_1^2p_2^2)$. Again, by Theorem1.3, the graph $\overline{MG(Z_{p_1^sp_2^t})}$ is non-planar.

If $t=1<s$, a similar K_5 subgraph appears, so the graph $\overline{MG(Z_{p_1^sp_2^t})}$ remains non-planar.

Proposition3.6

1. The graph $MG(Z_{p_1^sp_2^t})$ is Eulerian if and only if both s and t are odd.
2. The graph $\overline{MG(Z_{p_1^sp_2^t})}$ is non-Eulerian.

Proof

If s and t are odd numbers, then by Theorem3.1, every vertex in $MG(Z_{p_1^sp_2^t})$ has even degree. It follows from the Theorem1.4 that the graph $MG(Z_{p_1^sp_2^t})$ is Eulerian. Assume that at least one of s or t is even, say s. Then by Theorem3.1, we have $deg(p_2^2)=s+1$. In this case, $MG(Z_{p_1^sp_2^t})$ is not Eulerian.

Next, we show that the graph $\overline{MG(Z_{p_1^sp_2^t})}$ is non-Eulerian. Since $deg(p_1)=t$ and $deg(p_2)=s$ in $\overline{MG(Z_{p_1^sp_2^t})}$, the graph $\overline{MG(Z_{p_1^sp_2^t})}$ is not Eulerian when either s or t is odd.

Furthermore, since $deg\left(p_1^i\right)=st+s-3$ in $\overline{MG(Z_{p_1^sp_2^t})}$ for every $i=1,2,\ldots,s$, the graph $\overline{MG(Z_{p_1^sp_2^t})}$ is also not Eulerian when both s and t are even.

Theorem3.7

The graph $MG(Z_{p_1^sp_2^t})$ is Hamiltonian if and only if $s=t=2$.

Proof

If $s=1<t$, then (p_2) is an end vertex of $MG(Z_{p_1^sp_2^t})$. Hence $MG(Z_{p_1^sp_2^t})$ is non-Hamiltonian.

Suppose that $t>2=s$.

Then

$A=\{(p_1^ip_2^j): i=1,2,\ldots,s$ and $j=1,2,\ldots,t\}$

is an independent set of vertices in $MG(Z_{p_1^sp_2^t})$. Moreover, the vertices in A are adjacent only to (p_1) and (p_2). Therefore, $MG(z_{p_1^sp_2^t})$ does not contain a Hamiltonian cycle.

Similarly, $MG(z_{p_1^sp_2^t})$ is non-Hamiltonian when $s,t>2$.

If $s=t=2$, then $MG(Z_{p_1^sp_2^t})$ contains a Hamiltonian cycle:

$C:(p_1),(p_1^2),(p_1p_2^2),(p_2),(p_1^2p_2),(p_2^2),(p_1p_2),(p_1)$

In this case, $MG(Z_{p_1^sp_2^t})$ is Hamiltonian.

Proposition3.8

The graph $\overline{MG(Z_{p_1^sp_2^t})}$ is a connected graph if $(s,t)\notin\{(1,2),(2,1)\}$. Otherwise, $\overline{MG(Z_{p_1^sp_2^t})}$ has an isolated vertex.

Proof

If $(s,t)\in\{(1,2),(2,1)\}$, then (p_1p_2) is an isolated vertex in $\overline{MG(Z_{p_1^sp_2^t})}$.

It is easy to verify that $\overline{MG(Z_{p_1^sp_2^t})}$ is connected when $s=t=2$.

Now assume $s>t\geq 2$. Define the following sets of vertices:

$A_1=\{(p_1^ip_2): i=2,3,\ldots,s\}$,
$A_2=\{(p_1p_2^i): i=2,3,\ldots,t\}$,
$A_3=\{(p_1p_2)\}$,
$A_4=\{(p_1^i): i=2,3,\ldots,s\}$,
$A_5=\{(p_2^j): j=2,3,\ldots,t\}$,
$A_6=\{(p_2)\}$, $A_7=\{(p_1)\}$, and
$A_8=\{(p_1^ip_2^j): i=2,3,\ldots,s$ and $j=1,2,\ldots,t\}$.

Now, the vertices of $\overline{MG(Z_{p_1^sp_2^t})}$ in A_1 are mutually adjacent. Also, every vertex in A_1 is adjacent to every vertex in A_2 within $\overline{MG(Z_{p_1^sp_2^t})}$.

Similarly, the following diagram illustrates the adjacency relationships between pairs of vertices in $\overline{MG(Z_{p_1^s p_2^t})}$:

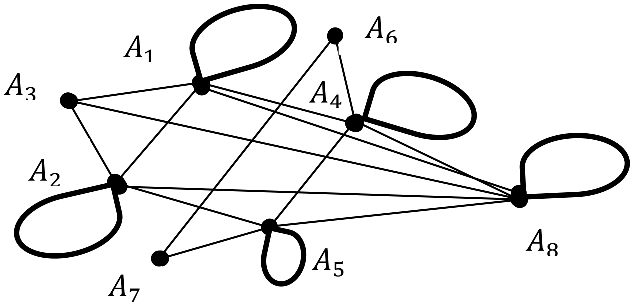

Figure3: The graph $\overline{MG(Z_{p_1^s p_2^t})}$

Thus, $\overline{MG(Z_{p_1^s p_2^t})}$ is connected. In the same way, we can show that $\overline{MG(Z_{p_1^s p_2^t})}$ is connected when $s = 1$ and $t > 2$.

Theorem3.9

Hosoya polynomial of $MG(Z_{p_1^s p_2^t})$ is

$$H\left(MG\left(Z_{p_1^s p_2^t}\right), x\right) = (0.5(s^2t^2 + s^2 + t^2) + s^2t + st^2 - 4.5st - 1.5(s+t) + 5)x^2 + (4st - 4)x + st + s + t - 1.$$

Proof

First, we have the polynomial

$$H(MG(Z_{p_1^s p_2^t}), x) = \sum_{k=0}^{2} d(MG(Z_{p_1^s p_2^t}), k)x^k$$
$$= a_o + a_1x + a_2x^2.$$

The order of $MG(Z_{p_1^s p_2^t})$ is

$$a_o = d\left(MG\left(Z_{p_1^s p_2^t}\right), 0\right) = st + s + t - 1.$$

By Corollary3.2, the size of $MG(Z_{p_1^s p_2^t})$ is

$$a_1 = d(MG(Z_{p_1^s p_2^t}), 1) = 4st - 4.$$

On the other hand, Lemma1.5 gives that:

$$a_2 = \frac{1}{2}a_o(a_o + 1) - a_o - a_1$$
$$= 0.5s^2t^2 + s^2t + st^2 - 4.5\,st + 0.5s^2 + 0.5t^2 - 1.5s + 1.5t + 5$$
$$= 0.5(s^2t^2 + s^2 + t^2) + s^2t + st^2 - 4.5st - 1.5(s+t) + 5$$

Thus,

$$H\left(MG\left(Z_{p_1^s p_2^t}\right), x\right) = (0.5(s^2t^2 + s^2 + t^2) + s^2t + st^2 - 4.5st - 1.5(s+t) + 5)x^2 + (4st - 4)x + st + s + t - 1.$$

4. Maximal Ideal Graph of $Z_{p_1^k p_2 p_3}$

In this section, some results of $MG(Z_{p_1^k p_2 p_3})$ will be obtained, where p_1, p_2 and p_3 are distinct prime numbers and k is a positive integer.

Theorem4.1

In the graph $MG\left(Z_{{p_1}^k p_2 p_3}\right)$,

$$deg(v) = \begin{cases} 4k-2 & if\ v = (p_1) \\ 2k & if\ v \in \{(p_2), (p_3)\} \\ 4 & if\ v \in \bigcup_{i=2}^{k}\{(p_1^i), (p_1^ip_2), (p_1^ip_3)\} \\ 2k+2 & if\ v \in \{(p_1p_2), (p_1p_3), (p_2p_3)\} \\ k+2 & if\ v = (p_1p_2\,p_3) \\ 3 & if\ v \epsilon \{(p_1^2p_2p_3), (p_1^3p_2p_3), \dots, (p_1^{k-1}p_2p_3)\} \end{cases}$$

Proof

The vertices adjacent to $(p_1p_2p_3)$ in $MG\left(Z_{{p_1}^k p_2 p_3}\right)$ are (p_2), (p_3) and (p_1^i) for $1 \le i \le k$.

Therefore, the total number of vertices adjacent to $(p_1p_2p_3)$ is $deg((p_1p_2p_3) = k + 2$.

In the same way, we can determine the degree of all other vertices in $MG(Z_{p_1^k p_2 p_3})$.

Corollary4.2

The size of $MG(Z_{p_1^k p_2 p_3})$ is

$$\left|E(MG(Z_{p_1^k p_2 p_3}))\right| = 15k - 6.$$

Proof

From Theorem1.2, we have

$$2\left|E\left(MG\left(Z_{p_1^k p_2 p_3}\right)\right)\right| = \mathrm{deg}(p_1) + deg(p_2) + deg(p_3) + deg(p_1p_2) + deg(p_1p_3) + deg(p_2p_3) + \sum_{i=2}^{k} deg(p_1^i) + \sum_{i=2}^{k} deg\left(p_1^ip_2\right) + \sum_{i=2}^{k} deg\left(p_1^ip_3\right) + deg(p_1p_2p_3) + \sum_{i=2}^{k-1} deg(p_1^ip_2p_3)$$

From Theorem4.1, we obtain

$$\left|E(MG(Z_{p_1^k p_2 p_3}))\right| = \frac{1}{2}((4k-2) + 4k + 3(2k+2) + 12(k-1) + (k+2) + 3(k-2)$$
$$= 15k - 6\,.$$

Theorem4.3

In the graph $MG(Z_{p_1^k p_2 p_3})$,

$$c\left(MG\left(Z_{p_1^k p_2 p_3}\right)\right) = \begin{cases} \left|MG(Z_{p_1^k p_2 p_3})\right| & if\ \ k = 1, 2 \\ 12 & if\ \ k \ge 3 \end{cases}.$$

Proof

The following cycles are the longest cycle in $MG(Z_{p_1 p_2 p_3})$ and $MG(Z_{p_1^2 p_2 p_3})$, respectively:

C_6: , $(p_1p_2), (p_1), (p_1p_3), (p_3), (p_2p_3), (p_2), (p_1p_2)$, and

C_{10}: $(p_1), (p_1^2), (p_1p_2), (p_1p_3), (p_1^2p_2), (p_2), (p_2p_3), (p_3), (p_1^2p_3), (p_3), (p_1p_2p_3), (p_1)$.

Then we have

$$c(MG(Z_{p_1^kp_2p_3})) = \left|MG(Z_{p_1^kp_2p_3})\right| \text{ for } k = 1, 2.$$

Now, suppose that $k \geq 3$. Then

C_{12}: $(p_1), (p_1^2), (p_1p_2p_3), (p_2), (p_1^2p_2p_3), (p_3), (p_1p_3), (p_1^2), (p_1p_2), (p_1^2p_3), (p_2p_3), (p_1^2p_2), (p_1)$

is a cycle in $MG(Z_{p_1^kp_2p_3})$.

By the definition of the maximal ideal graph, there is no cycle in $MG(Z_{p_1^kp_2p_3})$ with length greater than 12. Thus, $c(MG(Z_{p_1^kp_2p_3})) = 12$.

Proposition4.4

The graph $MG(Z_{p_1^kp_2p_3})$ is 4-partite graph.

Proof

We partition the vertex set of $MG(Z_{p_1^kp_2p_3})$ into the following sets:

$A_1 = \{(p_1), (p_2p_3)\}$,

$A_2 = \{(p_2) \cup \{(p_1^2p_3), (p_1^3p_3), \dots, (p_1^kp_3)\} \cup \{(p_1^2), (p_1^3), \dots, (p_1^k)\} \cup \{(p_1^2), (p_1^3), \dots, (p_1^k)\}$,

$A_3 = \{(p_3)\} \cup \{(p_1p_2), (p_1^2p_2), \dots, (p_1^kp_2)\}$, and

$A_4 = \{(p_1p_2p_3), (p_1^2p_2p_3), \dots, (p_1^{k-1}p_2p_3)\} \cup \{(p_1p_3)\}$.

Then no two vertices within the same A_i are adjacent in $MG(Z_{p_1^kp_2p_3})$ for every $i = 1, 2, \dots, 4$. Thus, $MG(Z_{p_1^kp_2p_3})$ is 4-partite graph.

Proposition4.5

1. The graph $MG(Z_{p_1^kp_2p_3})$ is Eulerian if and only if $k \in \{1, 2\}$.
2. The graph $\overline{MG(Z_{p_1^kp_2p_3})}$ is not Eulerian.

Proof

If $k = 1$, then by Theorem4.1, the degree of any vertex of $MG(Z_{p_1^kp_2p_3})$ is either 2 or 4.

Similarly, when $k = 2$, the degree of every vertex of $MG(Z_{p_1^kp_2p_3})$ is either 4 or 6. From both cases and Theorem1.4, we conclude that $MG(Z_{p_1^kp_2p_3})$ is Eulerian.

If $k \geq 3$, then $deg\,(p_1^3p_2p_3) = 3$. In this case, $MG(Z_{p_1^kp_2p_3})$ is not Eulerian.

Now, consider the graph $\overline{MG(Z_{p_1^kp_2p_3})}$.

We observe that

$$\begin{aligned} deg(p_2) &= \left(\left|\overline{MG\left(Z_{p_1^kp_2p_3}\right)}\right| - 1\right) - 2k \\ &= (4k + 2) - 1 - 2k \\ &= 2k + 1, \end{aligned}$$

which is an odd number. Thus, $\overline{MG(Z_{p_1^kp_2p_3})}$ is not Eulerian for every positive integer k.

Theorem4.6

The graphs $MG(Z_{p_1^kp_2p_3})$ is planar if and only if $k = 1$.

Proof

If $k = 1$, then the following figure shows that $MG(Z_{p_1p_2p_3})$ is planar:

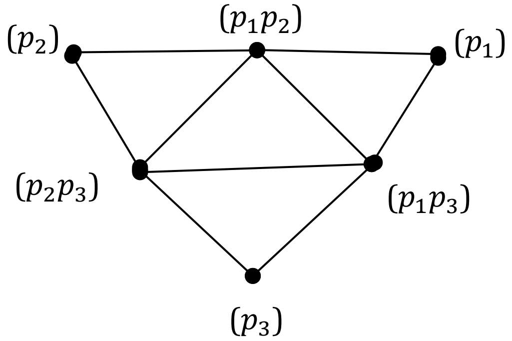

Figure 4: The graph $MG(Z_{p_1p_2p_3})$

Let $k\epsilon\{2, 3\}$.

Define

$V_1 = \{(p_1), (p_2), (p_3)\}$ and

$V_2 = \{(p_1p_2p_3), (p_1^2p_2), (p_1^2p_3)\}$.

Obviously, every vertex in $V_1$is adjacent to every vertex in V_2.

The vertices (p_2) and (p_3) are not adjacent to $(p_1^2p_3)$ and $(p_1^2p_2)$, respectively.

Additionally, (p_1p_2) is adjacent to both (p_2) and $(p_1^2p_3)$. Also, (p_1p_3) is adjacent to both $(p_1^2p_2)$ and (p_3) . Then the induced subgraph of $MG(Z_{p_1^kp_2p_3})$ by the vertex set

$\{(p_1), (p_2), (p_3), (p_1p_2p_3), (p_1^2p_2), (p_1^2p_3), (p_1p_2), (p_1p_3)\}$

is a subdivision of $K_{3,3}$. Then by Theorem1.3, the graph $MG(Z_{p_1^kp_2p_3})$ is non-planar.

If $k \geq 4$, then the graph $MG(Z_{p_1^kp_2p_3})$ contains a $K_{3,3}$ with partite sets:

$V_1 = \{(p_1), (p_2), (p_3)\}$ and

$V_2 = \{(p_1p_2p_3), (p_1^2p_2p_3), (p_1^3p_2p_3)\}$.

Again, by Theorem1.3, the graph $MG(Z_{p_1^kp_2p_3})$ is non-planar.

Theorem4.7

Hosoya polynomial of the graph $MG(Z_{p_1^kp_2p_3})$ is

$$H\left(MG\left(Z_{p_1^kp_2p_3}\right), x\right) = (8k^2 - 9k + 7)x^2 + (15k - 6)x + (4k + 2)$$

Proof

First, we have the polynomial

$H(MG(Z_{p_1^kp_2p_3}), x) = \sum_{k=0}^{2} d(MG(Z_{p_1^kp_2p_3}), k)x^k$

$$=a_o+a_1x+a_2x^2.$$

The order of $MG(Z_{p_1^kp_2p_3})$ is

$$a_o=d\left(MG\left(Z_{p_1^kp_2p_3}\right),0\right)=4k+2.$$

By Corolary4.2, the size of $MG(Z_{p_1^kp_2p_3})$ is

$$a_1=d(MG(Z_{p_1^kp_2p_3}),1)=15k-6.$$

On the other hand, Lemma1.5 gives that

$$\begin{aligned}a_2&=\frac{1}{2}a_o(a_o+1)-a_o-a_1\\&=\frac{1}{2}((4k+2)^2+4k+2)-(4k+2)-\\&\quad(15k-6)\\&=8k^2-9k+7\end{aligned}$$

Thus,

$$H\left(MG\left(Z_{p_1^kp_2p_3}\right),x\right)=(8k^2-9k+7)x^2+(15k-6)x+(4k+2)$$

5. Conclusion

The study of the maximal ideal graph of the ring Z_n provides valuable insights into both the algebraic and graph-theoretic structure of the ring. Since Z_n is a finite commutative ring with unity, its ideal structure is completely determined by the divisors of n. This paper investigates the structure, graph properties, and characterizations of the maximal ideal graph of the ring Z_n.

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