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Reducing Impulse Noise in Images Using an Improved Formula Conjugate Gradient Method

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ABSTRACT

The conjugate formula's significance is frequently emphasised by conjugate gradient approaches. In this paper, a novel conjugate coefficient for the conjugate gradient technique is introduced using a quadratic model and conjugacy condition. This coefficient is used to address image restoration issues. The algorithms described in this study exhibit the essential descent property and global convergence. The new approach is far better, as shown by numerical experiments. It has been demonstrated that the innovative conjugate gradient strategy outperforms the conventional FR conjugate gradient method. The great improvement of the new method has been confirmed by numerical testing. It has been demonstrated that the novel conjugate gradient approach outperforms the widely used FR approach.

1.Introduction

Impulse noise from noisy sensors or communication systems frequently taints images. Impulse noise is represented by two primary models. The first type of noise is salt-and-pepper noise, where noisy pixels can only accept the maximum and minimum

values of pixels in the original image's dynamic range. Random-valued noise is another type of impulsive noise, where the noisy pixels can have any random value between the dynamic range's maximal and minimal pixel values. One of the most significant issues in picture processing is getting rid of the noise mentioned above. There are two common approaches for eliminating impulsive noise for this purpose: (1) When the noise ratio is large, the median filter and its variations, may identify noisy pixels but do a poor job of restoring them. Uncorrupted pixels maintain their grey levels. The restored image can be warped and lose some of its features. (2) The variational technique can effectively preserve edges and features, but it alters the grey level of all pixels, including those that are uncorrupted (see, Basim, 2022).

The purpose of this study is to provide a set of iterative techniques for optimization problems where the objective function is edge-preserving regularization (EPR) functionals. Using two equations that anticipate which pixels are likely to be contaminated, an adaptive median filter (AMF) (Cai et al., 2007) identifies and filters impulsive noise. $A = \{1, 2, 3, \dots, M\} \times \{1, 2, 3, \dots, N\}$ is the true representation, and is the index set of X .

The objective is to identify the pixels in a photograph that contain noise. The first stage in the procedure is to determine the indices of the detected noise pixels, sometimes referred to as $W \subset A$. Furthermore, the group of four pixels that are adjacent to the pixel at location $(i, j) \in A$ is indicated by $P_{i,j}$. Furthermore, $y_{i,j}$ denotes the observed value of the pixel at location (i, j) , in the picture. Finally, $u_{i,j} = [u_{i,j}]_{(i,j) \in N}$ is a lexicographically ordered column vector of length N . The number of elements in N is represented by the letter c . c is the letter that represents the number of elements in N , see (XUE et al., 2018). To recover noise pixels, the equation minimizes the following functional:

$$f_{\alpha}(u) = \sum_{(i,j) \in N} \left[|u_{i,j} - y_{i,j}| + \frac{\beta}{2} (2 \times S_{i,j}^1 + S_{i,j}^2) \right]$$

the regularization parameter is indicated by β and $S_{i,j}^1 = 2 \sum_{(m,n) \in P_{i,j} \cap N^c} \phi_{\alpha}(u_{i,j} - y_{m,n})$, $S_{i,j}^2 = \sum_{(m,n) \in P_{i,j} \cap N} \phi_{\alpha}(u_{i,j} - y_{m,n})$. By minimizing a smooth functional without the requirement for a non-smooth data-fitting component, $\phi_{\alpha} = \sqrt{\alpha + x^2}, \alpha > 0$ is a possible function that may be utilized to minimize noise in an optimization process:

$$f_{\alpha}(u) = \sum_{(i,j) \in N} [(2 \times S_{i,j}^1 + S_{i,j}^2)] \tag{2}$$

For more information the image processing process is illustrated in the following diagram:

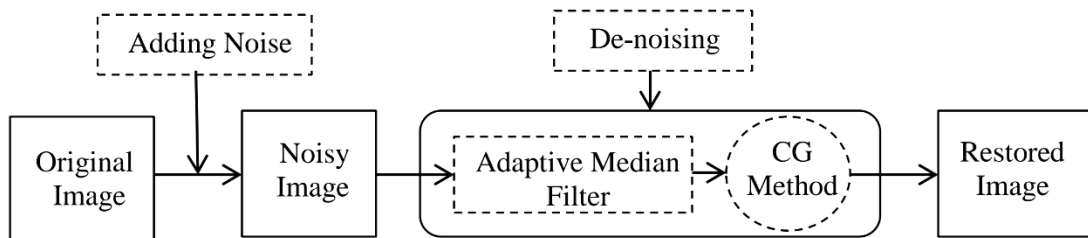


Figure 1.1. Flow chart of our experiments

The conjugate gradient method is a key part of image restoration. An issue is often expressed as:

$$\text{Min}f_{\alpha}(u) , u \in R^n \tag{3}$$

$f_{\alpha}(u): R^n \rightarrow R$ is the smooth function. One kind of iterative technique that produces a sequence with the desired structure is the conjugate gradient method as:

$$u_{k+1} = u_k + \alpha_k d_k \tag{4}$$

where d_k is the search direction and α_k is the typical step length discovered by an acceptable precise line search, as:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k} \tag{5}$$

See (Wright, 2006). The parameter α_k represents the step length as following, which is defined by Wolfe conditions:

$$f(u_k + \alpha_k d_k) \leq f(u_k) + \delta \alpha_k g_k^T d_k \tag{6}$$

$$d_k^T g(u_k + \alpha_k d_k) \geq \sigma d_k^T g_k \tag{7}$$

where $0 < \delta < \sigma < 1$, see (Jiang and Jian, 2013). The conjugate-gradient technique may be used to determine the search direction using the following formula:

$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{8}$$

where β_k is a scalar. There are two methods to solve this, called the Fletcher-Reeves (FR) method (Fletcher, 1964) and the Dai-Yuan (DY) method (Dai and Yuan, 1999). These methods can be represented as follows:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \beta_{k+1}^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k} \tag{9}$$

Many studies have been carried out regarding the convergence characteristics that conjugate gradient methods exhibit (Dai *et al.*, 2000; Hager *et al.*, 2005). The FR technique converges worldwide when accurate line searching is employed, as demonstrated by Zoutendijk (programming and 1970, n.d.), who initiated the study. Several scholars have so far developed a distinct conjugate gradient coefficient that converge to the global solution and exhibit outstanding numerical performance. Recent studies on the conjugate-gradient approach may

be found in Basim's (Abbas Hassan, 2019) and Hideaki and Yasushi's (Iiduka and Narushima, 2012), where the parameters are specified as follows:

$$\beta_k^{HY} = \frac{\|g_{k+1}\|^2}{2/\alpha_k(f_k - f_{k+1})}, \beta_k^B = \frac{\|g_{k+1}\|^2}{(f_k - f_{k+1})/\alpha_k - g_k^T d_k/2} \tag{10}$$

These algorithms effectively achieve all of the features and benefits, as demonstrated in (Polak *et al.*, n.d.). It is also quite efficient in terms of computation. The improved performance is said to be better for unconstrained tasks and maximizes the advantages of the original quadratic model conjugate gradient approaches. Additional sources regarding the CG technique can be found at (A. Hassan and Ahmed A. Abdullah, 2022; Basim A. Hassan and Hameed M. Sadiq, 2022; Hassan and Sadiq, 2022).

Convergence analysis and the conjugate gradient approach follow, starting with a new derivation of the coefficient conjugate and new viewpoint on the denominator $d_k^T G v_k$ based on the quadratic model. Afterward, the numerical outcomes of the suggested methodology are given and contrasted with those of other conjugate gradient methods.

1. Conjugating a new formula

The Taylor series is where we begin in order to derive the new formula:

$$f(x) = f(x_{k+1}) - g_{k+1}^T s_k + \frac{1}{2} s_k^T Q(x_{k+1}) s_k \tag{11}$$

where Q is Hessian matrix. The derivation of equation (11) for s_k is:

$$g_{k+1} = g_k + Q(u_{k+1}) s_k \tag{12}$$

By applying (12) to (11) as:

$$s_k^T Q(u_k) s_k = 2(f_k - f_{k+1}) + 2y_k^T s_k + 2g_k^T s_k \tag{13}$$

Thus, using some algebra, and by using (12) and (13), which obtained:

$$s_k^T Q(u_k) s_k = 1/2 y_k^T s_k + (f_{k+1} - f_k) - g_k^T s_k \tag{14}$$

In the search direction d_{k+1} , using conjugate condition $d_{k+1}^T y_k = 0$ will yield:

$$\beta_k = \frac{g_{k+1}^T Q s_k}{d_k^T Q s_k} \tag{15}$$

Putting (14) into (15), it can be observed that:

$$\beta_k = \frac{g_{k+1}^T y_k}{1/2 y_k^T d_k + (f_{k+1} - f_k)/\alpha_k - g_k^T d_k} \tag{16}$$

It reduces to the following using the exact line

search in (16):

$$\beta_k = \frac{\|g_{k+1}\|^2}{1/2y_k^T d_k + (f_{k+1} - f_k)/\alpha_k - g_k^T d_k} \tag{17}$$

As a result, our idea, which is known as the BBG.

Algorithm BBG.

Initialization. Given $x_0 \in R^n$, set $k = 0$, $d_0 = -g_0$.

Stage 1. When $\|g_k\| \leq \varepsilon$ then stop.

Stage 2. Utilizing (6a) and (6b), compute α_k .

Stage 3. Compute β_k using (17), assuming

$$x_{k+1} = x_k + \alpha_k d_k.$$

Stage 4. Determine $d_{k+1} = -g_{k+1} + \beta_k d_k$.

Stage 5. Proceed to step 2 after setting $k=k+1$.

2. Global convergence:

We examine the global convergence features of the method in this section. First, the following is established:

1. On level set $\Omega = \{u: u \in R^n, f(u) \leq f(u_1)\}$, $f(u)$ is bounded.
2. Since the $\nabla f(u)$ is Lipschitz continuous with $L > 0$, the following inequality is true for every $\tau, v \in R^n$:
3. $\|g(\tau) - g(v)\| \leq L\|\tau - v\|, \forall \tau, v \in R^n$ (18)

Theorem 1.

The directions of Algorithm BBG are descent.

Proof :

Used the definition β_k^{BBG} in equation (17) and multiply by g_{k+1} , we get:

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{1/2y_k^T d_k + \frac{f_{k+1} - f_k}{\alpha_k} - g_k^T d_k} d_k^T g_{k+1} \tag{19}$$

which leads us to:

$$d_{k+1}^T g_{k+1} = \frac{d_k^T g_{k+1} - 1/2y_k^T d_k + (f_{k+1} - f_k)/\alpha_k - g_k^T d_k}{1/2y_k^T d_k + (f_{k+1} - f_k)/\alpha_k - g_k^T d_k} \|g_{k+1}\|^2 \tag{20}$$

which means:

$$d_{k+1}^T g_{k+1} = \frac{\|g_{k+1}\|^2}{1/2y_k^T d_k + (f_{k+1} - f_k)/\alpha_k - g_k^T d_k} g_k^T d_k \tag{21}$$

Then:

$$d_{k+1}^T g_{k+1} = \beta_k^{BBG} g_k^T d_k < 0 \tag{22}$$

This finishes the proof.

Understanding the Zoutendijk condition (programming and 1970, n.d.), or the lemma, is crucial to analyzing the convergence qualities of the CG approach.

Lemma1.

Given that (1) and (2) hold true, α_k meets the Wolfe criteria, and d_k is the direction of descent, then:

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty \tag{23}$$

Theorem 2.

Relation Wolfe conditions may be used to derive the step length α_k . For BBG method with descent directions d_{k+1} we get:

$$\liminf_{k \rightarrow \infty} \|g_{k+1}\| = 0 \tag{24}$$

Proof :

Assume, by contradiction, that equation (24) is false. Next, we may identify a $r > 0$ for each k such that:

$$\|g_{k+1}\| > r \tag{25}$$

However, after squaring both sides and using search time as $d_{k+1} + g_{k+1} = \beta_k d_k$, we get:

$$\|d_{k+1}\|^2 + \|g_{k+1}\|^2 + 2d_{k+1}^T g_{k+1} = (\beta_k)^2 \|d_k\|^2 \tag{26}$$

Making use of (22) to (26) suggests:

$$\|d_{k+1}\|^2 = \frac{(d_{k+1}^T g_{k+1})^2}{(d_k^T g_k)^2} \|d_k\|^2 - 2d_{k+1}^T g_{k+1} - \|g_{k+1}\|^2 \tag{27}$$

Upon dividing (27) by $(d_{k+1}^T g_{k+1})^2$, we get at:

$$\begin{aligned} \frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} &= \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \frac{\|g_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} - \frac{2}{d_{k+1}^T g_{k+1}} \leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \left(\frac{\|g_{k+1}\|}{(d_{k+1}^T g_{k+1})} + \frac{1}{\|g_{k+1}\|^2} \right) + \frac{1}{\|g_{k+1}\|^2} \\ &\leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} + \frac{1}{\|g_{k+1}\|^2} \end{aligned} \tag{28}$$

Thus, we arrived at:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} \leq \sum_{i=1}^{k+1} \frac{1}{\|g_i\|^2} \tag{29}$$

Assume that for every $k \in n$, there exists $c_1 > 0$ such that $\|g_k\| \geq c_1$ exists. Then:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k+1}{c_1^2} \tag{30}$$

At last, we obtain:

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = \infty \tag{31}$$

Equivalently, as stated in Lemma 1, $\liminf_{k \rightarrow \infty} \|g_k\| = 0$ is holds.

3. Numerical experiments

We offer numerical data that demonstrates the efficiency of New in minimizing salt-and-pepper compulsion noise. We evaluate the efficiency of both the New method and the FR technique. All simulations are done on a PC with MATLAB 2017a. These are the conditions that will result in the termination of all techniques:

$$\begin{aligned} \frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} &\leq 10^{-4} \text{ and } \|f(u_k)\| \leq \\ &10^{-4}(1 + |f(u_k)|) \end{aligned} \tag{32}$$

Along with the exam text, the test images feature Lena, House, the Cameraman, and Elaine. The test photograph is also included. To evaluate the quality of the restoration, we utilize the PSNR (peak signal to noise ratio) in a similar way to (XUE *et al.*, 2018; Yu *et al.*, 2010). Inspired by earlier works (A. Hassan and A. Alashoor, 2023; Hassan and Alashoor, 2022), this paper presents some update. The subsequent explanation pertains to the assessment of the restoration's

effectiveness:

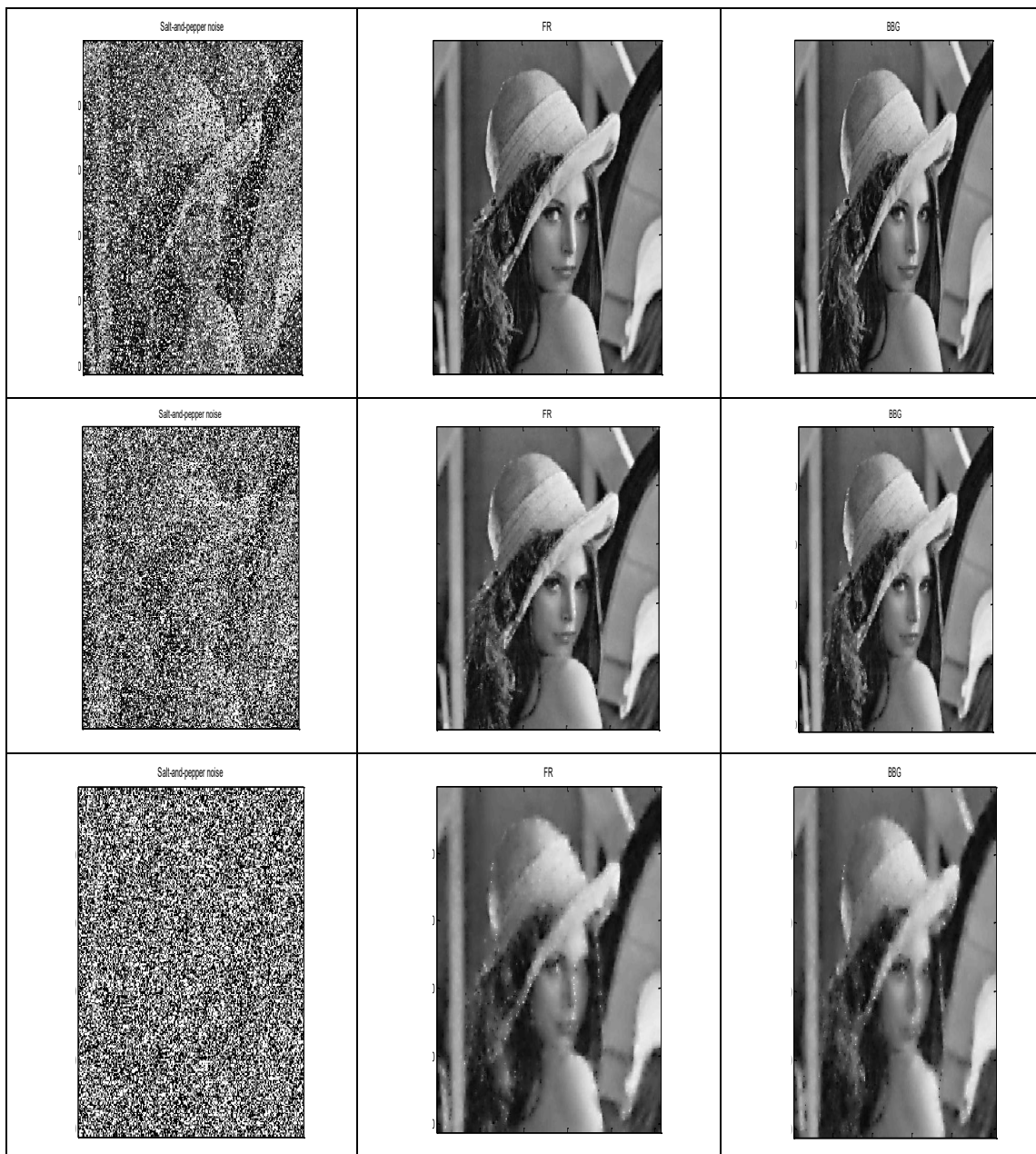
$$PSNR = 10 \log_{10} \frac{255^2}{\frac{1}{MN} \sum_{i,j} (u_{i,j}^r - u_{i,j}^*)^2} \tag{33}$$

The pixel values of the original image are denoted as $u_{i,j}^r$, while the restored image's pixel values are denoted as $u_{i,j}^*$. Three metrics including number of iterations (NI), number of function evaluation (NF), and PSNR are used to measure the efficiency of all algorithms. This research illustrates how many function evaluations and iterations are required to complete the denoising process and the image's PSNR following the use of this method. The new approach is more efficient than the FR methodology in terms of speed, as the latter takes a significant amount of time to complete. Supporting data can be found in Table 1. The novel strategy yields PSNR values that are similar to those obtained using the FR method.

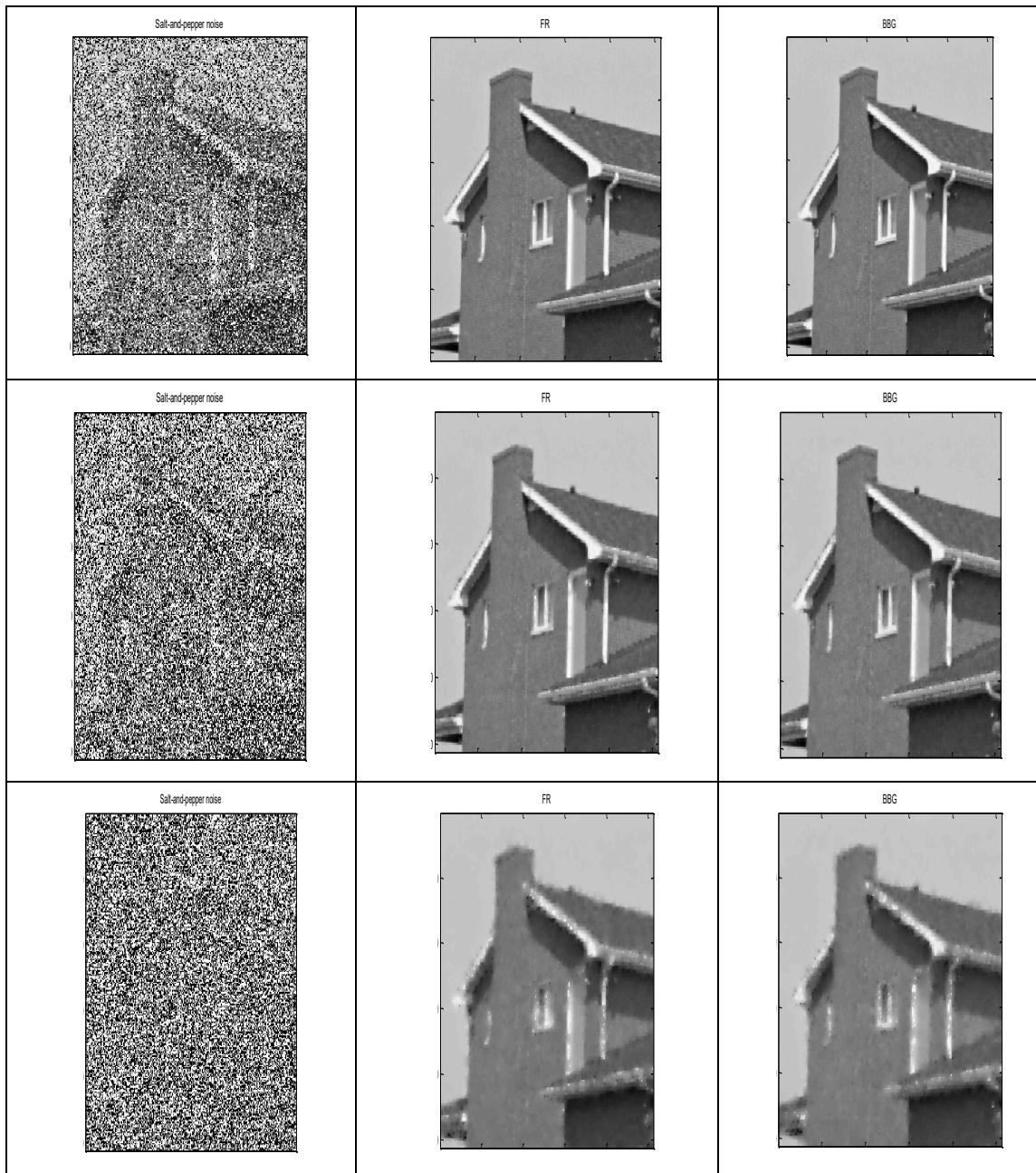
Table1. Numerical. Results. of BBG and FR algorithms.

.Image	Noise level r (%)	FR..Method			BBG..Method		
		.NI	.NF	P.SNR (d.B)	.NI	.NF	P.SNR (d.B)
Le	50	82	153	30.5529	66	69	30.4113
	70	81	155	27.4824	74	77	27.4298
	90	108	211	22.8583	97	182	22.6163
Ho	50	52	53	30.6845	44	45	34.7005

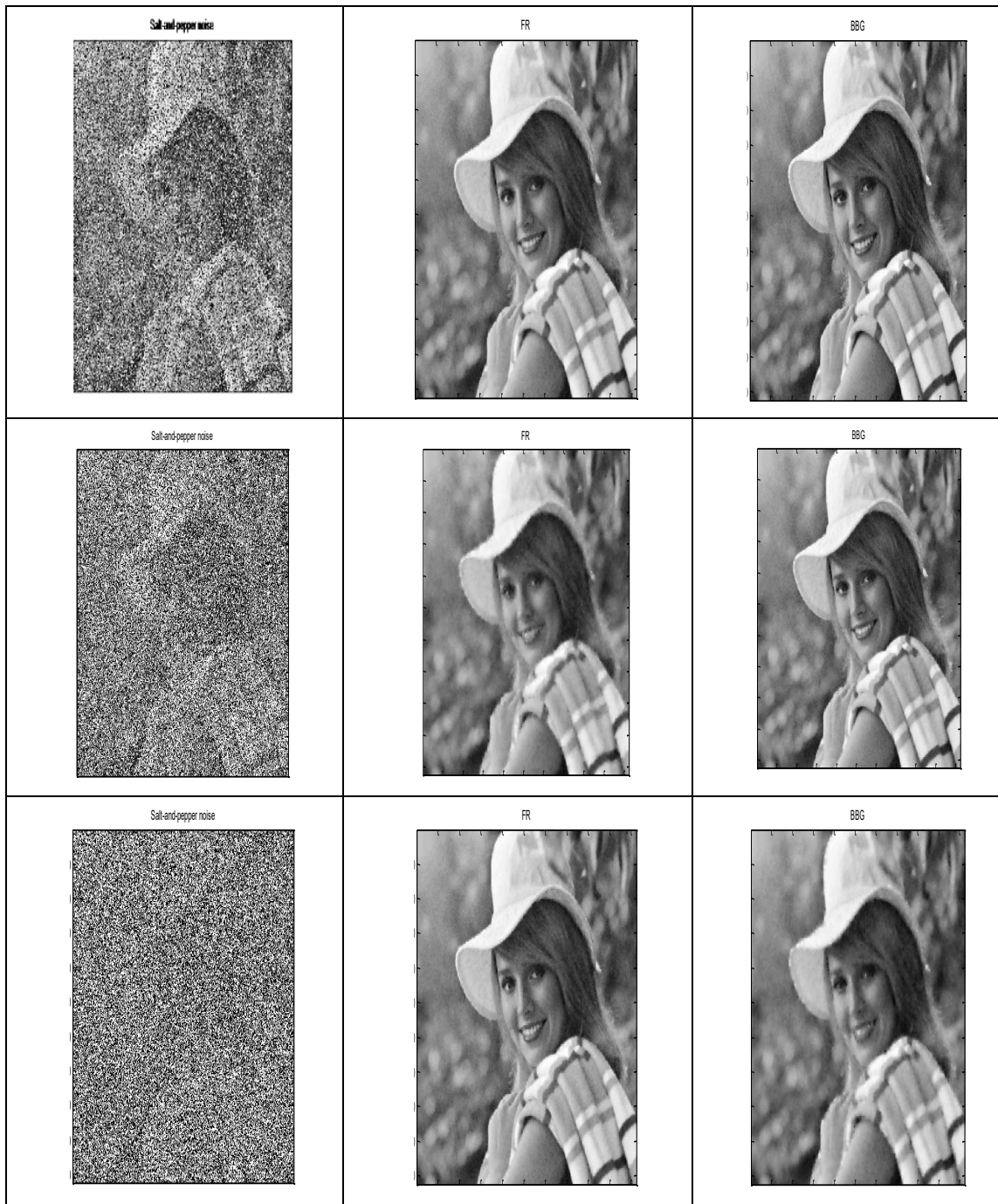
	70	63	116	31.2564	48	49	30.9161
	90	111	214	25.287	70	73	25.3304
EI	50	35	36	33.9129	32	33	33.8054
	70	38	39	31.864	38	39	31.8949
	90	65	114	28.2019	55	57	28.171
c512	50	59	87	35.5359	39	47	35.5545
	70	78	142	30.6259	45	51	30.482
	90	121	236	24.3962	78	138	24.8642



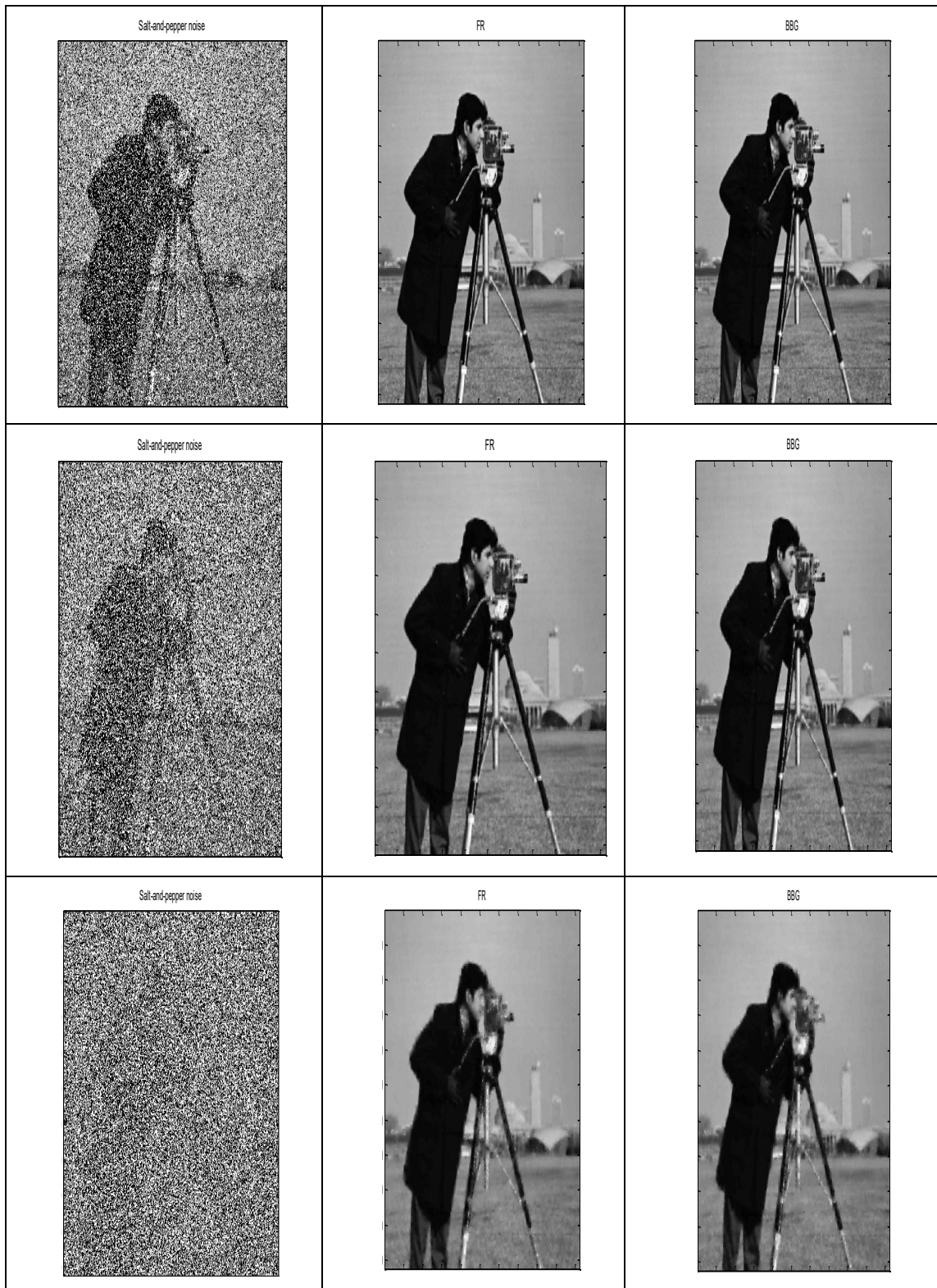
The outcomes of the FR and BBG algorithms applied to the 256 * 256 Lena image are illustrated in **Figure 1**.



The outcomes of the FR and BBG algorithms applied to the 256 * 256 House image are illustrated in **Figure 2**.



The outcomes of the FR and BBG algorithms applied to the 256 * 256 Elaine image are illustrated in **Figure 3**.



The outcomes of the FR and BBG algorithms applied to the 256 * 256 Cameraman image are illustrated in **Figure 4**.

4. Conclusions

A recently discovered formula was also reviewed, along with many conjugate gradient

techniques, such as BBG. We used search criteria to determine the global convergence of the Wolfe line. It has been demonstrated

that these techniques, and BBG in particular, may effectively reduce the number of function evaluations and simulation iterations without compromising the quality of the visual result.

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