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The Effect of the Density of Square-Free ω_p-numbers on the Bounds of Beurling Counting Function

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ABSTRACT

Primitive weird numbers are weird numbers which are not a multiple of any smaller weird numbers. The goal of this work is to use a square-free primitive weird number $x = a\prod_{i=1}^{n} q_i$ where $\{q_i\}_{i=1}^{n}$ be an increasing sequence of prime numbers such that q_1 is greater than $\prod_{j=1}^{r}(\bar{q}_j + 1)$ and $a = \prod_{j=1}^{r} \bar{q}_j$ is deficient number with n greater than 1, to enhancing the classic bounds of Beurling counting function.

1.Introduction

Let x be a natural number, and $\sigma(x)$ be the sum of its divisors. Then x is called abundant and deficient if $\overline{d}(x) = \sigma(x) - 2x > 0,$ $-\overline{d}(x) = \underline{d}(x) = 2x - \sigma(x) > 0$ respectively. The number x is called primitive abundant number (PA-number) if $\overline{d}(x) > 0$ but for $a|x$, $\underline{d}(a) > 0$. We say x is weird (ω-number) if x is abundant and which cannot be express as a sum of distinct proper divisors. We call a ω-number x primitive (ω_p-number) if it is not a multiple of any smaller ω-number. This introduction also addresses some details from lectures about Beurling prime system and the generalized Chebyshev's function $\Psi_{\mathfrak{z}}(x)$ which would be the integer part of x minus 1. That is:

$$\Psi_{\mathfrak{z}}(x) = x + O(1)$$

Where the Beurling prime system is very closed to being discrete system ($N_{\mathfrak{z}}$ is a step function) and investigating how regular the corresponding generalized counting function of integers $N_{\mathfrak{z}}$ to be. Now we move our attention to address the bounds of $\left(N_{\mathfrak{z}}(x) - \tau x\right)$ which has been investigated by number of writers for example (Al-Maamori and Hilberdink, 2015) as follows:

On the Riemann Hypothesis the upper bound for $\left|N_{\mathfrak{z}}(x) - \tau x\right|$ can be improved to:

$$(1.1) \qquad \left|N_{\mathfrak{z}}(x) - \tau x\right| \le Dx\, exp\left\{-\frac{d \log x \log_3 x}{\log_2 x}\right\},$$

for every $d < 0.25$ and $D > 0$.

Furthermore, they also find the lower bound for $\left|N_{\mathfrak{z}}(x) - \tau x\right|$, they have:

$$(1.2) \qquad \left|N_{\mathfrak{z}}(x) - \tau x\right| \ge Bx\ exp\left\{\frac{-b \log x \log_4 x}{\log_3 x}\right\},$$

for every $b > 1$ and $B > 0$.This shows that there is a small gap between these results which reflects the great difficulty in determining the behaviors of $\zeta_{\mathfrak{z}}(\sigma + it)$ in the strip $0.5 < \sigma < 1$. In a previous paper (Hasan et al., 2018) we have been restricted the gap between the lower bound of $(N_{\mathfrak{z}} - \tau x)$ and the upper bound of $(N_{\mathfrak{z}} - \tau x)$ on R.H. and we get:

$$(1.3) \qquad \left|N_{\mathfrak{z}}(x) - \tau x\right| \ge Axe^{-ak_x}$$

for every $a > 1$ and $A > 0$, where

$$k_x = \log x \sqrt{\frac{\log_4 x}{\log_3 x}}.$$

The aim of this work is to use the form of the square-free ω_p-numbers to refinement the upper and the lower bounds of $(N_{\mathfrak{z}} - \tau x)$.

The following notation will be standard in every part of this work:

- $\Delta(x) = \left|N_{\mathfrak{z}}(x) - \tau x\right|$.
- $\log_2 x = \log\log x$, $\log_3 x = \log\log\log x$.
- $y = \log_3 x$, $z = \log_4 x$.
- $q, q_i, \overline{q_i}$ are primes.
- q_* largest prime divisor of x.

2.Basic ideas

In this section we recall some results of generating ω_p-number from square-free prime numbers, so we start with:

Lemma 2.1:

Let a be a square-free deficient number $(a = \prod_{j=1}^{r} \bar{q}_j)$ and there exists a prime $q > \overline{q_r}$ such that aq is abundant, then for $n > 0$ there are an increasing sequence of prime numbers $\{q_i\}_{i=1}^{n}$ such that $b = \prod_{i=1}^{n} q_i$ with $q_1 > \overline{q_r}$, $x = ab$ is abundant and $a\prod_{i=1}^{s} q_i$ is deficient for all $s < n$. Hence x is PA-number.

This **result is proved in** (Hasan et al., 2019).

Proposition 2.2:

Let a be a square-free deficient number $(a = \prod_{j=1}^{r} \bar{q}_j)$ and n greater than 1. Let $\{q_i\}_{i=1}^{n}$ be an increasing sequence of prime numbers such that $b = \prod_{i=1}^{n} q_i$ with q_1 is greater than $\prod_{j=1}^{r}(\bar{q}_j + 1)$. Let

$$\ell = flow\left(\frac{q_1 - \prod_{j=1}^{r}(\bar{q}_j + 1)}{q_n - q_1}\right).$$

Let $x = ab$ and

$$\mathcal{K} = \cup_{k=0}^{\ell} \left\{c \in \mathbb{N} | k q_n + \prod_{j=1}^{r}(\bar{q}_j + 1) < c < (k + 1)q_1\right\}.$$

Then for all $c \in \mathcal{K}$, c cannot be expressible as a sum of distinct divisors of x.

This **result is proved in** (Hasan et al., 2019).

Here for completeness, we will repeat the proof of the following theorem that was proved in (Hasan et al., 2019).

Theorem 2.3:

Let a be a square-free deficient number $(a = \prod_{j=1}^{r} \bar{q}_j)$ and there exists a prime $q > \overline{q_r}$ such that aq is abundant. For n greater than 1, let $\{q_i\}_{i=1}^{n}$ be an increasing sequence of prime numbers such that $b = \prod_{i=1}^{n} q_i$ with q_1 is greater than $\sigma(a)$. If $x = ab$ is abundant and $\overline{d}(x)$ belongs to $\mathcal{K}$, then x is ω_p-number.

Proof:

Since a number is ω_p-number iff it's ω-numbers and PA-numbers by Proposition 4.1 in (Amato et al., 2019).

First, since $\overline{d}(x)$ cannot be express as a sum of distinct divisors of x by proposition 2.2, hence x is ω-number by using lemma 2 in (Melfi, 2015).

The second part of the proof is to show that x is PA-number it is enough to show that $\overline{d}(a \prod_{i=1}^{s} q_i) < 0$ for all $s < n$.

If $n = 2$, then

$$\begin{aligned}\overline{d}(aq_1) &= \sigma\big(\textstyle\prod_{j=1}^{r} \bar{q}_j\big)(q_1 + 1) - 2\prod_{j=1}^{r} \bar{q}_j\, q_1 \\ &= \big(\textstyle\prod_{j=1}^{r}(\bar{q}_j + 1) - 2\prod_{j=1}^{r} \bar{q}_j\big)q_1 + \prod_{j=1}^{r}(\bar{q}_j + 1) \\ &= \overline{d}\big(\textstyle\prod_{j=1}^{r} \bar{q}_j\big)q_1 + \prod_{j=1}^{r}(\bar{q}_j + 1)\end{aligned}$$

So $\overline{d}(aq_1)$ less than zero (deficient), since $\overline{d}(a) \leq -1$ and q_1 greater than $\prod_{j=1}^{r}(\bar{q}_j + 1)$.

Now, we went to show that $\overline{d}(aq_1 \ldots q_{n-1}) < 0$ for n greater than or equal 3. We have

$$\begin{aligned}\overline{d}(a \textstyle\prod_{i=1}^{n-1} q_i) &= \overline{d}\left(\frac{a \prod_{i=1}^{n} q_i}{q_n}\right) \\ &= \sigma\left(\frac{a \prod_{i=1}^{n} q_i}{q_n}\right) - 2\,\frac{a \prod_{i=1}^{n} q_i}{q_n} \\ &= \frac{\sigma(a \prod_{i=1}^{n} q_i)}{q_n + 1} - 2\,\frac{a \prod_{i=1}^{n} q_i}{q_n} \\ &= \frac{\overline{d}(a \prod_{i=1}^{n} q_i) - 2\frac{a \prod_{i=1}^{n} q_i}{q_n}}{q_n + 1}.\end{aligned}$$

Therefore, $\overline{d}(a \prod_{i=1}^{n-1} q_i) < 0$ since for $\overline{d}(x) \in \mathcal{K}$, so

$\overline{d}(x) < (k + 1)q_1 \leq (\ell + 1)q_1 < (\frac{q_1}{n} + 1)q_1 < q_1^2$.

And

$$2\,\frac{a \prod_{i=1}^{n} q_i}{q_n} = 2a \prod_{i=1}^{n-1} q_i > 2aq_1^2 > q_1^2.$$

So $\overline{d}(a \prod_{i=1}^{n} q_i) < 2a \prod_{i=1}^{n-1} q_i$.

Hence $a \prod_{i=1}^{s} q_i$ is deficient for all $s < n$.

As $x = a \prod_{i=1}^{n} q_i$ is abundant and by using lemma 2.1, so x is PA-number. Hence x is ω_p-number.

3.Enhancing the bounds of $\Delta(x)$

In (Apostol, 1976) Theorem (4.7) proved the following result. For k greater than or equal 1 the k –prime q_k satisfies the inequalities

$$\text{(1.4)} \qquad \frac{1}{6} k \log k < q_k < 12(k \log k + k \log 12e^{-1}).$$

We will use this result to enhancing the bounds of $\Delta(x)$ in (1.1), (1.2) and (1.3). In this paper, we improve it to the following.

Theorem 3.1:

For x goes to infinity, then

$$\text{(1.5)} \qquad Bx\, exp\left\{\frac{-\gamma \log x}{\log_3 x}\right\} \leq \Delta(x) \leq Dx\, exp\left\{\frac{-\delta \log x}{\log_2 x}\right\}$$

where $B, D > 0$.

Here γ, δ denotes positive constant greater than 1. The method of proof, for lower and upper bounds of Beurling counting function is a refinement of the one in (Al-Maamori and Hilberdink, 2015) by using the properties of a square-free ω_p-numbers.

Proof of the upper bound on R.H.: For suppose that $x = ab$ is such a square-free ω_p-number, where $a = \prod_{i=1}^{r} \overline{q_i}\,, b = \prod_{i=1}^{k} q_i$ with $q_i < q_{i+1}$ for all i, the largest prime factors of b is grater than $\frac{1}{6} k \log k$ and the smallest prime factors of b is less than $\prod_{i=1}^{r}(\overline{q_i} + 1)$. Then $\bar{d}(x) > 0$, but $\underline{d}(a) < 0$. So

$$\begin{aligned}\log x &= \log a + \log b \\ &= \textstyle\sum_{i=1}^{r} \log \overline{q_i} + \sum_{i=1}^{k} \log q_i \\ &> \textstyle\sum_{i=1}^{k} \log q_i \\ &> \log q_* \\ &> \log(\tfrac{1}{6} k \log k) = D_1.\end{aligned}$$

For given positive value of D_1 fixed. Then

$$y = \log_3 x = \log_2(\log x) > \log_2 D_1.$$

And so,

$$\frac{y}{e^{y}} > \frac{\log_2 D_1}{e^{y}}$$

Therefore, the exponetial term in upper bound became:

$$exp\left\{\frac{-\beta y \log x}{e^{y}}\right\} < exp\left\{\frac{-\delta \log x}{e^{y}}\right\}.$$

For given positive value of δ fixed. Thus

$$\Delta(x) \le Dx\, exp\left\{\frac{-\delta \log x}{\log_2 x}\right\}.$$

Proof of the lower bound: to prove the lower bound in (1.5), we may suppose that $x = \prod_{i=1}^{k} q_i$ is such a square-free ωp-number,

if $q_1 < q_2 < \cdots < q_k$ are any k primes less than $12(k \log k + k \log 12e^{-1})$.Then $\bar{d}(x) > 0$,

but $\underline{d}(\prod_{i=1}^{r} q_i) < 0$ for all r less than k. For

$$z = \log y = \log_4 x = \log_3\left(\log \prod_{i=1}^{k} q_i\right)$$

$$= \log_3(\textstyle\sum_{i=1}^{k} \log q_i).$$

But $\log q_i < \log q_*$ for all i and so,

$$\textstyle\sum_{i=1}^{k} \log q_i < k \log q_* < kq_*.$$

But $q_* < 12(k \log k + k \log 12e^{-1})$. Then we have

$$z < \log_3 kq_*$$

$$< \log_2(\log(12(k \log k + k \log 12e^{-1}))) = E$$

The optimal choice of E is greater than zero. Consequently, the exponetial term becomes

$$exp\left\{\frac{-\beta z \log x}{y}\right\} > exp\left\{\frac{-\beta E \log x}{y}\right\}.$$

i.e.

$$\Delta(x) \ge Bx\, exp\left\{\frac{-\gamma \log x}{\log_3 x}\right\}.$$

For positive constant γ. Thus we have proved that, for sufficiently large x, the result in (1.5) is follows.

Theorem 3.2:

For x goes to infinity, then

$$\Delta(x) \ge A\left(x e^{-A_1 \log x (\log_3 x)^{-\frac{1}{2}}}\right).$$

With A, A_1greater than 1.

Proof:

As for lower bound, we still consider only ωp-number (not greater than x) of the form $x = \prod_{i=1}^{k} q_i$, with $q_1 < q_2 < \cdots < q_k$ are any k primes less than $12(k \log k + k \log 12e^{-1})$ such thatn $\bar{d}(x) > 0$, but $\underline{d}(\prod_{i=1}^{r} q_i) < 0$ for all r less than k. Using (1.3) we get that:

$$\Delta(x) \ge Ax\, e^{-a \log x \left(\frac{\log y}{y}\right)^{\frac{1}{2}}}.$$

Now,

$$\log y = \log_4 x = \log_3 \textstyle\sum_{i=1}^{k} \log q_i.$$

But

$$kq_* > k \log q_k > \textstyle\sum_{i=1}^{k} \log q_i.$$

And in fact, from (1.4),

$$q_* < 12(k \log k + k \log 12e^{-1}).$$

Hence

$$\log y < \log_3(kq_*)$$
$$< \log_3(12k(k \log k + k \log 12e^{-1})).$$

So,

$$\Delta(x) \ge Ax\, e^{-a \log x \left(\frac{\log_3(12k(k \log k + k \log 12e^{-1}))}{y}\right)^{\frac{1}{2}}}$$

$$\ge Ax\, e^{-A_1 \log x \left(\frac{1}{y}\right)^{\frac{1}{2}}}.$$

for every A, A_1 greater than 1.

Thus, we have proved that

$$\Delta(x) \ge A\left(x e^{-A_1 \log x (\log_3 x)^{-\frac{1}{2}}}\right).$$

Conclusion

This work explains how could the form of ωp-numbers effect on the bound of the generalized counting function of integers $N_{\mathfrak{z}}$ and thus restrict the gap between the lower bound of $(N_{\mathfrak{z}}(x) - \tau x)$ and the upper bound of $(N_{\mathfrak{z}}(x) - \tau x)$ on Riemann Hypothesis. One can use this form to show the effect on another bounds of $\left(N_{\mathfrak{z}}(x) - \tau x\right)$.

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References

Al-Maamori, F. & Hilberdink, T. 2015. An example in Beurling's theory of generalised primes. *Acta Arithmetica,* 4, 383-395.

Amato, G., Hasler, M. F., Melfi, G. & Parton, M. 2019. Primitive abundant and weird numbers with many prime factors. *Journal of number theory,* 201**,** 436-459.

Apostol, T. M. 1976. Introduction to Analytic Number Theory. Springer-Verlag.

Hasan, S., Al-Maamori, F. & Abdulrahman, H. 2018. Restricted the gap between the error terms of Ω-results for (N_3− тx) and the error terms of O-results for (N− тx) on Riemann Hypothesis. *International Journal of Pure and Applied Mathematics,* 120**,** 751.

Hasan, S. S., Al-Maamori, F. & Majeed, L. 2019. A further restricting the gap between (N_3-тx) and (N-тx) on R. H. by using the sense of ω-numbers and ωp-numbers. *Journal of Advanced Research in Dynamical and Control Systems,* 11**,** 2043-2051.

Melfi, G. 2015. On the conditional infiniteness of primitive weird numbers. *Journal of Number Theory,* 147**,** 508-514.