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RESEARCHPAPER

# A theoretical investigation of $\mathcal{S}$-Numerical Range with the respect to a family of projections

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**A B S T R A C T:**

The idea of the $\mathcal{S}$-numerical range of a bounded linear operator on a complex Hilbert space with respect to a family of projections is introduced in this study. We provide a detailed description and discuss its relationship to the $\mathcal{S}$-numerical range and generalizations such as product $\mathcal{S}$-numerical range. The significance of this new concept comes from its unifying character.

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## 1.INTRODUCTION:

Let $\mathcal{A} \in M_n$ be the algebra of $n \times n$ complex matrices and $\mathcal{S} \in M_n$ be a self-adjoint operator, both defined on a finite complex Hilbert space $\mathbb{C}^n$. Then we consider the $\mathcal{S}$-numerical range and the positive and negative $\mathcal{S}$-numerical ranges denoted by

$$W_{\mathcal{S}}(\mathcal{A}) = \left\{\frac{\langle \mathcal{S}\mathcal{A}\psi, \psi\rangle}{\langle \mathcal{S}\psi, \psi\rangle} : \psi \in \mathbb{C}^n and \ \langle \mathcal{S}\psi, \psi\rangle = 1\right\} \quad (1)$$

and

$$W_{\mathcal{S}}^{\pm}(\mathcal{A}) = \left\{\frac{\langle \mathcal{S}\mathcal{A}\psi, \psi\rangle}{\langle \mathcal{S}\psi, \psi\rangle} : \psi \in \mathbb{C}^n and \ \langle \mathcal{S}\psi, \psi\rangle = \pm 1\right\}. \quad (2)$$

respectively. Where $W_{\mathcal{S}}^{+}(\mathcal{A})$ is the set of positive $\mathcal{S}$-numerical range and $W_{\mathcal{S}}^{-}(\mathcal{A})$ is the set of negative $\mathcal{S}$-numerical range of an operator $\mathcal{A}$, which have been studied by other researchers(K. Li, N.K. Tsing, F. Uhlig, 1996),(R.D. Grigorieff, R. Platto, 1995). [1]

The sets $W_{\mathcal{S}}^{\pm}(\mathcal{A})$ generalize the well-known and widely used notation of classical numerical range.

$$W(\mathcal{A}) = \{\langle \mathcal{A}\xi, \xi\rangle : \xi \in \mathbb{C}^n \text{and } \|\xi\| = 1\}. \quad (3)$$

Which introduced by Toeplitz in (Toeplitz, 1918) that is a practical tool for studying operator matrices and operators, has been extensively examined. As an extensive background for numerical range and its properties we refer to see,(Wlat Hamad, Ahmed Muhammad, 2020),(Dirr, Gunther, and Frederik vom Ende, 2020)and reference therein. We also introduce the set $\mathcal{S}$-numerical range for a bounded linear

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operator $\mathcal{A}$ and a self-adjoint operator $\mathcal{S}$ in the infinite dimensional case as follows:

$$W_{\mathcal{S}}(\mathcal{A}) = \left\{\frac{\langle \mathcal{S}\mathcal{A}\psi, \psi\rangle}{\langle \mathcal{S}\psi, \psi\rangle} : \psi \in Hand\ \langle \mathcal{S}\psi, \psi\rangle = 1\right\}. \quad (4)$$

The $S$-numerical ranges generalize the classical numerical range and some properties of the $W(\mathcal{A})$ can be extended to $W_{\mathcal{S}}(\mathcal{A})$ as follows: Consider $\mathcal{A} \in \mathbb{M}_n$, where $\mathbb{M}_n$ be the algebra of $n \times n$ complex matrices and $\mathcal{S} \in \mathbb{M}_n$ be Hermitian matrix. The set $W_{\mathcal{S}}^{\pm}(\mathcal{A})$ is well-known set in which each of the sets $W_{\mathcal{S}}^{+}(\mathcal{A})$ and $W_{\mathcal{S}}^{-}(\mathcal{A})$ are convex sets. The relationships between the sets $W_{\mathcal{S}}^{+}(\mathcal{A})$ and $W_{\mathcal{S}}^{-}(\mathcal{A})$ are described in (Bebiano, N., Lemos, R., Da providencia, J. and Soares, 2005) and (Nakazato,H., Bebiano, N. & D A Providencia, J., 2011). According to Bayasgalan (Bayasgalan, 1991), the set $W_{\mathcal{S}}^{+}(\mathcal{A})$ is convex if $\mathcal{S}$ is nonsingular and indefinite. Although sharing some analogous properties with the classical numerical range, has a quite different behavior. Unlike the numerical range $W_{\mathcal{S}}(\mathcal{A})$ is not convex. One easily checks that $W_{-\mathcal{S}}^{+}(\mathcal{A}) = W_{\mathcal{S}}^{-}(\mathcal{A})$, so is $W_{\mathcal{S}}(\mathcal{A}) = W_{\mathcal{S}}^{+}(\mathcal{A}) \cup -W_{-\mathcal{S}}^{+}(\mathcal{A})$. If $\mathcal{S} = I_n$ then $W_{\mathcal{S}}^{-}(\mathcal{A})$ is the empty set and the set $W_{\mathcal{S}}(\mathcal{A})$ reduce to classical numerical range, see ((Bebiano, N., Lemos, R., Da providencia, J. and Soares, 2005), and reference therein).Furthermore, (Bebiano, N., Lemos, R., Da providencia, J. and Soares, 2005), shows that $\sigma(\mathcal{A}_P) \subset \overline{W_{\mathcal{S}}(\mathcal{A})}$ if $\mathcal{A}$ is positive definite. More generally, the following properties are known:

1. $W_{\mathcal{S}}(\mathcal{A}) = W_{\mathcal{S}}(U^*\mathcal{A}U)$for any finite matrix $U$ and any nonsingular Hermitian operator $\mathcal{S}$ such that $U^*\mathcal{S}U = \mathcal{S}$. Also,$W_{\mathcal{S}}(\alpha\mathcal{A} + \beta\mathcal{S}) = \alpha W_{\mathcal{S}}(\mathcal{A}) + \beta$, for any $\alpha, \beta \in \mathbb{C}$.
2. It is clear that $W_{\mathcal{S}}(\mathcal{A}^*) = \overline{W_{\mathcal{S}}(\mathcal{A})}$ where $\mathcal{A}^*$ is self-adjoint operator.
3. For any operators $\mathcal{A}$ and $\mathcal{B}$ we have $W_{\mathcal{S}}(\mathcal{A} + \mathcal{B}) \subset W_{\mathcal{S}}(\mathcal{A}) + W_{\mathcal{S}}(\mathcal{B})$.
4. $W_{\mathcal{S}}(\mathcal{A})$, it may not be closed and is either unbounded or a singleton (Bebiano, N., Lemos, R., Da providencia, J. and Soares, 2005).
5. For any $\lambda \in \mathbb{C}, W_{\mathcal{S}}(\mathcal{A}) = \{\lambda\}$ if and only if $\mathcal{S}\mathcal{A} = \mathcal{S}\lambda$ and we have $W_{\mathcal{S}}(\mathcal{A}) \subset \mathbb{R}$ if and only if $\mathcal{A}$ is hermitian.

The great advantage of the $\mathcal{S}$-numerical range, when compared to the spectrum, is that it is relatively easy to compute (certainly in the case of matrices). It became an effective tool in numerous physics(N. Bebiano, J. Da Providˆencia, 1998) and reference therein, applications as well as in numerous disciplines of pure and applied mathematics, including control theory (E. Rogers, K. Galkowski, and D.-H. Owens, 2007) and operator theory(C.-K. Li and Y.-T. Poon, , 2011),(Berivan Faris Azeez , Ahmed Muhammad, 2020). Moreover, numerical range with respect to a family of projections has been investigated by (Waed D., Joachim K. and Nazife E. Ö., 2018), and we generalize this concept to $\mathcal{S}$-numerical range with respect to a family of projections due to its definition there are interesting connections between the $\mathcal{S}$-numerical range and $\mathcal{S}$-numerical range with respect to a family of projections which have been discussed.

The following is the structure of this paper. Section 2 is devoted to the main definition of $\mathcal{S}$-numerical range with the respect to a family of projetions and some basic remarks. In section 3.1, we are going to establish the connection of family of projections with the $\mathcal{S}$-numerical range. In section 3.2 we will define the product $\mathcal{S}$-numerical range and establish connection to the product $\mathcal{S}$-numerical range that plays a notable effect in quantum information theory (Gawron P, Puchała Z, Miszczak JA, Skowronek Ł, Życzkowski K, 2010)(Waed D., Joachim K. and Nazife E. Ö., 2018),(Gawron, Piotr. "Z. Pucha la, JA Miszczak, L. Skowronek, K. Zyczkowski., 2011). Also we will assume that the underlying Hilbert space $\mathcal{H}$ is given as a tensor product of two (separable) Hilbert spaces $\mathcal{H}_k$ and $\mathcal{H}_l$ where $\mathcal{H}$ is finite-dimensional of composite dimension $n = kl$ where $dim\,\mathcal{H}_k = k$ and $dim\,\mathcal{H}_l = l$.

## 2. $\mathcal{S}$-Numerical range with respect to a family of projections:

We are interested in the definition of the $\mathcal{S}$-numerical range of a bounded linear operator $\mathcal{A}$ with respect to families of orthogonal projections. For $k \in \mathbb{N}$ and a bounded self-adjoint operator $\mathcal{S}$ we define an operator $\mathcal{S}\mathcal{A}_P$ on the range $ran(P)$ by

$$\mathcal{SA}_P: ran(P) \to ran(P), x \mapsto \mathcal{SA}_P x := P\mathcal{SA}x.$$

Where$\mathcal{P}_k := \{P \in \mathbb{P}: dim(ran(P)) = k\}$,

and

$\mathbb{P} := \{ P \in \mathcal{L}(\mathcal{H}): P\ is$ orthogonal projection in $\mathcal{H}$ }.

**Remark 2.1** 1. The sets $\mathbb{P}$,$\mathcal{P}_k$ are closed with the operator norm, (Waed D., Joachim K. and Nazife E. Ö., 2018).
2.The relation between $\mathcal{SA}_P$ and $\mathcal{SA}$ are expressed by

$$\mathcal{SA}_P P = P\mathcal{SA}P.$$

where $\mathcal{SA}_P$ is called the compression of $\mathcal{SA}$ to $ran(P)$ and $\mathcal{SA}$ is called a dilation of $\mathcal{SA}_P$ to $\mathcal{H}$.

**Proposition 2.2** Let $\mathcal{A}$ be a bounded linear operator and $\mathcal{S}$ be a self-adjoint operator both on a complex Hilbert space $\mathcal{H}$, then

$$W_{\mathcal{S}}(\mathcal{A}_P) \subset W_{\mathcal{S}}(\mathcal{A})$$

Proof. Suppose that $\lambda \in W_{\mathcal{S}}(\mathcal{A}_P)$ then there are $f \in ran(P)$ with $\langle \mathcal{S}f, f\rangle \neq 0$ and $Pf = f$, thus

$$\lambda = \frac{\langle \mathcal{SA}_P f, f\rangle}{\langle \mathcal{S}f, f\rangle} = \frac{\langle P\mathcal{SA}Pf, f\rangle}{\langle \mathcal{S}f, f\rangle} = \frac{\langle \mathcal{SA}Pf, Pf\rangle}{\langle \mathcal{S}f, f\rangle} = \frac{\langle \mathcal{SA}f, f\rangle}{\langle \mathcal{S}f, f\rangle}. \quad (5)$$

We conclude that $\lambda \in W_{\mathcal{S}}(\mathcal{A})$.

**Definition 2.3** Let $\mathcal{A}$ be a bounded linear operator and $\mathcal{S}$ be a self-adjoint operator both on a complex Hilbert space $\mathcal{H}$ and $\mathcal{P} \subseteq \mathbb{P}$. Then we define

$$W_{\mathcal{S},\mathcal{P}}(\mathcal{A}) := \bigcup_{P\in\mathcal{P}} \frac{\langle f, f\rangle}{\langle \mathcal{S}f, f\rangle} \sigma(\mathcal{SA}_P) \quad (6)$$

for which $\langle \mathcal{S}f, f\rangle \neq 0$,is called the $\mathcal{S}$-numerical range of $\mathcal{A}$ with respect to a family of orthogonal projections $\mathcal{P}$.

**Remark 2.4**

1. For a bounded linear self-adjoint operator $\mathcal{A}$ one obtains $W_{\mathcal{S},\mathcal{P}}(\mathcal{A}) \subseteq \mathbb{R}$, where $(\mathcal{SA})^* = \mathcal{SA}$.

   Let $\mathcal{A}$ be self-adjoint operator, as we know the set $\sigma(\mathcal{SA}_P)$ contains all eigenvalues of the form $P\mathcal{SA}Pf = \lambda f$ then after simple calculation and fixed $P \in \mathcal{P}_k$. For$\lambda \in W_{\mathcal{S},\mathcal{P}}(\mathcal{A})$we have the following equalities:

$$\lambda = \frac{\langle P\mathcal{SA}Pf, f\rangle}{\langle \mathcal{S}f, f\rangle} = \frac{\langle f, P^*\mathcal{A}^*\mathcal{S}^*P^*f\rangle}{\langle f, \mathcal{S}^*f\rangle} = \frac{\langle f, P^*(\mathcal{SA})^*P^*f\rangle}{\langle f, \mathcal{S}f\rangle} = \left(\frac{\langle f, P\mathcal{SA}Pf\rangle}{\langle f, \mathcal{S}f\rangle}\right) = \left(\frac{\overline{\langle P\mathcal{SA}Pf, f\rangle}}{\overline{\langle \mathcal{S}f, f\rangle}}\right) = \overline{\left(\frac{\langle P\mathcal{SA}Pf, f\rangle}{\langle \mathcal{S}f, f\rangle}\right)}.$$

   Therefor we obtain that $\lambda \in \mathbb{R}$.

2. $W_{\mathcal{S},\mathcal{P}}(\mathcal{A}^*) = (W_{\mathcal{S},\mathcal{P}}(\mathcal{A}))^*$, where $\mathcal{SA}^* = \mathcal{A}^*\mathcal{S}$.

   The properies of inner product space and adjoint give us $\left(\frac{\langle P\mathcal{SA}Pf, f\rangle}{\langle \mathcal{S}f, f\rangle}\right)^* = \frac{\langle P\mathcal{A}^*\mathcal{S}^*Pf, f\rangle}{\langle \mathcal{S}f, f\rangle}$.

   Then we have $\left(\frac{\langle P\mathcal{SA}Pf, f\rangle}{\langle \mathcal{S}f, f\rangle}\right)^* = \frac{\langle P\mathcal{SA}Pf, f\rangle}{\langle \mathcal{S}f, f\rangle}$.

   Consequently$W_{\mathcal{S},\mathcal{P}}(\mathcal{A}^*) = (W_{\mathcal{S},\mathcal{P}}(\mathcal{A}))^*$.

## 3. *Main Results*

### *3.1 Connection to the $\mathcal{S}$-numerical range.*

In this section we establish the connection of family of projections with the $\mathcal{S}$-numerical range. For the proof note that, for each $P \in \mathcal{P}_k$ and for any orthogonormal basis $\{f_i\}_{i=1}^k$ of range of the projection $P$ and we denoted by $ran(P)$, one has

$$Px = \sum_{i=1}^{k} \frac{\langle x, f_i\rangle}{\langle f_i, f_i\rangle} f_i \qquad \forall x \in \mathcal{H}. \quad (7)$$

**Theorem 3.1** *Let $\mathcal{A}$ be a bounded linear operator and $\mathcal{S}$ be a self-adjoint operator both on a complex Hilbert space $\mathcal{H}$, then*

$$W_{\mathcal{S},\mathcal{P}_1}(\mathcal{A}) = W_{\mathcal{S}}(\mathcal{A}).$$

*Proof.* Suppose $\lambda \in W_{\mathcal{S},\mathcal{P}_1}(\mathcal{A})$. Then there are $P \in \mathcal{P}_1$ and $f \in ran(P)$ with $\langle \mathcal{S}f, f\rangle \neq 0$, for which $\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle} P\mathcal{S}\mathcal{A}Pf = \lambda f$. Therefore

$$\langle \lambda f, f\rangle = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\langle P\mathcal{S}\mathcal{A}Pf, f\rangle.$$

Hence $\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\langle \mathcal{S}\mathcal{A}Pf, Pf\rangle = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\langle \mathcal{S}\mathcal{A}f, f\rangle$.

Therefore $\lambda = \frac{\langle \mathcal{S}\mathcal{A}f,f\rangle}{\langle \mathcal{S}f,f\rangle}$,

implies that $\lambda \in W_{\mathcal{S}}(\mathcal{A})$.

Conversly: Suppose $\lambda \in W_{\mathcal{S}}(\mathcal{A})$. Then there exists $f \in \mathcal{H}$ with $\langle \mathcal{S}f, f\rangle \neq 0$ such that $\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle} P\mathcal{A}\mathcal{S}Pf$. We assume that $P$ denote the orthogonal projection onto span$\{f\}$. Then by Eq.(7), we have

$$\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle} P\mathcal{S}\mathcal{A}Pf = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle} P\mathcal{S}\mathcal{A}f = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\frac{\langle \mathcal{S}\mathcal{A}f,f\rangle f}{\langle f,f\rangle} = \lambda f,$$

therefore $\lambda \in W_{\mathcal{S},\mathcal{P}}(\mathcal{A})$.

The following result is a generalization of Theorem 3.1

**Proposition 3.2** *Let $\mathcal{A}$ be a bounded linear operator and $\mathcal{S}$ be a self-adjoint operator both on a complex Hilbert space $\mathcal{H}$, and the family $\mathcal{P}_k$ with $k \in \mathbb{N}$, the following holds:*

1. If $\dim \mathcal{H} = k$, then $W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A}) = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma(\mathcal{S}\mathcal{A})$.
2. If $\dim \mathcal{H} < \infty$, then $W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A})$ is closed for $1 \leq k \leq \dim \mathcal{H}$.
3. If $\dim \mathcal{H} > k$, then $W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A}) = W_{\mathcal{S}}(\mathcal{A})$.

*Proof.* First case follows from the fact that, in the finite dimensional the set of all eigenvalues is equal to the spectrum of an operator $\mathcal{A}$, so $W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A}) = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma(\mathcal{S}\mathcal{A})$, given $\dim \mathcal{H} = k$. In order to proof second part, we remark that the cases $k = 1$ and $k = dim\, \mathcal{H}$ are readily covered by Theorem 3.1 and part 1. For proof of the other cases, we assume that $(\lambda_m)_{m\in\mathbb{N}} \subseteq W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A})$ with $\lambda_m \to \lambda \in \mathbb{C}$. Since $\lambda_m \in W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A})$ there exists $P_m \in \mathcal{P}_k$ and $f_m \in ran(P)$ with $\langle \mathcal{S}f, f\rangle \neq 0$ such that $\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle} P_m\mathcal{S}\mathcal{A}P_m f_m = \lambda_m f_m$. Since the Hilbert space is finite-dimensional, we conclude that

$$\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle} P\mathcal{S}\mathcal{A}Pf = \lim_{m\to\infty}\left(\frac{\langle f_m,f_m\rangle}{\langle \mathcal{S}f_m,f_m\rangle} P_m\mathcal{S}\mathcal{A}P_m f_m\right) = \lim_{m\to\infty}(\lambda_m f_m) = \lambda f \quad (8)$$

fo some (normalized) $f \in \mathcal{H}$ and $\lim_{m\to\infty} P_m := P \in \mathcal{P}_k$. Consequently, $\lambda \in W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A})$. In order to proof third case, we assume $\lambda \in W_{\mathcal{S}}(\mathcal{A})$ be given. Then there exists $f_0 \in \mathcal{H}$ with $\langle \mathcal{S}f_0, f_0\rangle \neq 0$ for which $\lambda = \frac{\langle \mathcal{S}\mathcal{A}f_0,f_0\rangle}{\langle \mathcal{S}f_0,f_0\rangle}$. Now take $f_1, f_2, \dots, f_{k-1} \in \mathcal{H}$ with $\langle \mathcal{S}f_i, f_i\rangle \neq 0$ for $0 \leq i \leq n$, such that $f_l \perp f_m$, and $f_l \perp \mathcal{S}\mathcal{A}f_0$ where $0 \leq l \neq m \leq k-1$. Let $P$ be the orthogonal projection onto $span\{f_0, f_1, \dots, f_{k-1}\}$ which is a $k$-dimensional subspace then, employing Eq.(3),

$$\frac{\langle f_0,f_0\rangle}{\langle \mathcal{S}f_0,f_0\rangle} P\mathcal{A}Pf_0 = \frac{\langle f_0,f_0\rangle}{\langle \mathcal{S}f_0,f_0\rangle} P\mathcal{A}f_0$$
$$= \frac{\langle f_0,f_0\rangle}{\langle \mathcal{S}f_0,f_0\rangle}\frac{\langle \mathcal{A}f_0,f_0\rangle f_0}{\langle f_0,f_0\rangle} + \frac{\langle f_0,f_0\rangle}{\langle \mathcal{S}f_0,f_0\rangle}\frac{\langle \mathcal{A}f_0,f_1\rangle f_1}{\langle f_1,f_1\rangle} + \cdots$$
$$+ \frac{\langle f_0,f_0\rangle}{\langle \mathcal{S}f_0,f_0\rangle}\frac{\langle \mathcal{A}f_0,f_{k-1}\rangle f_{k-1}}{\langle f_{k-1},f_{k-1}\rangle} = \frac{\langle \mathcal{S}\mathcal{A}f_0,f_0\rangle f_0}{\langle \mathcal{S}f_0,f_0\rangle} = \lambda f_0. \quad (9)$$

Consequently $\lambda \in W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A})$. Now suppose $\lambda \in W_{\mathcal{S},\mathcal{P}_k}(\mathcal{A})$. Then there exist $P \in \mathcal{P}_k$ and $f \in ran(P)$ with $\langle \mathcal{S}f, f\rangle \neq 0$, such that

$$\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle} P\mathcal{S}\mathcal{A}Pf = \lambda f.$$

Hence

$$\langle \lambda f, f\rangle = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\langle P\mathcal{S}\mathcal{A}Pf,f\rangle$$
$$= \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\langle \mathcal{S}\mathcal{A}f,f\rangle$$

Implie $\lambda = \frac{\langle \mathcal{S}\mathcal{A}f,f\rangle}{\langle \mathcal{S}f,f\rangle}$ and $\lambda \in W_{\mathcal{S}}(\mathcal{A})$.

In the next result we show how the family of projections is related to the point spectrum of the operator $\mathcal{S}\mathcal{A}$ and we define the set

$\mathcal{P}_{\mathcal{A}} := \{P \in \mathbb{P}: P\mathcal{S}\mathcal{A} = \mathcal{A}\mathcal{S}P = \mathcal{S}\mathcal{A}P, dim(ran(P)) < \infty\}$.

**Theorem 3.3** *Let $\mathcal{A}$ be a symmetric bounded linear operator and $\mathcal{S}$ be a self-adjoint both on a complex Hilbert space $\mathcal{H}$, then*

$W_{\mathcal{S},\mathcal{P}_{\mathcal{A}}}(\mathcal{A}) = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_P(\mathcal{S}\mathcal{A}_P)$, *where* $\langle \mathcal{S}f,f\rangle \neq 0$.

*Proof.* Suppose that $\lambda \in \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{S}\mathcal{A}_P)$ there exists $f \in \mathcal{H}$ such that

$$\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\mathcal{S}\mathcal{A}f = \lambda f \qquad (10)$$

Now, choose $P$ to be the orthogonal projection onto $span\{f\}$. Applying $P$ to the eigenvalue equation directly yields $\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}P\mathcal{S}\mathcal{A}Pf = \lambda Pf = \lambda f$ which shows that $\lambda$ is an eigenvalue of $P\mathcal{S}\mathcal{A}P$. On the other hand, for $x \in \mathcal{H}$,

$$P\mathcal{S}\mathcal{A}x = \frac{\langle \mathcal{S}\mathcal{A}x,f\rangle f}{\langle f,f\rangle} = \frac{\langle x,\mathcal{A}\mathcal{S}f\rangle f}{\langle f,f\rangle}$$

From Eq.(8) we have

$$P\mathcal{S}\mathcal{A}x = \frac{1}{\langle f,f\rangle}\langle x, \frac{\langle \mathcal{S}f,f\rangle}{\langle f,f\rangle}\lambda f\rangle f$$
$$= \frac{\langle \mathcal{S}f,f\rangle}{\langle f,f\rangle}\frac{\langle x,f\rangle\lambda f}{\langle f,f\rangle}$$
$$= \mathcal{S}\mathcal{A}\frac{\langle x,f\rangle f}{\langle f,f\rangle}$$
$$= \mathcal{S}\mathcal{A}Px.$$

This implies that $P \in \mathcal{P}_{\mathcal{A}}$.Therefore$\lambda \in W_{\mathcal{S},\mathcal{P}_{\mathcal{A}}}(\mathcal{A})$. Conversely: We assume $\lambda \in W_{\mathcal{S},\mathcal{P}_{\mathcal{A}}}(\mathcal{A})$. Then there exist a normalized $P \in \mathcal{P}_{\mathcal{A}}$ and $0 \neq f \in ran(P)$ with $\langle \mathcal{S}f,f\rangle \neq 0$ for which

$$\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}P\mathcal{S}\mathcal{A}Pf = \lambda f$$

and

$$\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}P\mathcal{S}\mathcal{A}Pf = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\mathcal{S}\mathcal{A}PPf = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\mathcal{S}\mathcal{A}Pf.$$

Since $f \in ran(P)$ we obtain

$$\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\mathcal{S}\mathcal{A}Pf = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\mathcal{S}\mathcal{A}f = \lambda f$$

and hence $\lambda \in \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{S}\mathcal{A})$.

**Remark 3.4** In theorem 3.3 when $\mathcal{A}$ is non-symmetric operator, then $\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_P(\mathcal{S}\mathcal{A})$ does not contain in $W_{\mathcal{S},\mathcal{P}_{\mathcal{A}}}(\mathcal{A})$.

**Theorem 3.5** *Let $\mathcal{A}$ be a bounded linear operator and $\mathcal{S}$ be a self adjoint operator both on a complex Hilbert space $\mathcal{H}$. Then*

$$W_{\mathcal{S},\mathcal{P}_{\mathcal{A}^*}}(\mathcal{A}) = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{A}^*) = (\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{A}))^*,$$

*where* $\mathcal{P}_{\mathcal{A}^*} := \{P \in \mathbb{P}: P\mathcal{A} = \mathcal{A}^*P,\ dim(ran(P)) < \infty\}$.

*Proof.* In view of Theorem 3.3 , we see $W_{\mathcal{S},\mathcal{P}_{\mathcal{A}^*}}(\mathcal{A}) = \frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{A}^*)$ and also note that $\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{A}^*) = (\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{A}))^*$ holds for all normal operators, it remaining to show that $P\mathcal{S}\mathcal{A} = \mathcal{A}^*\mathcal{S}P$. For any $x \in \mathcal{H}$, we have

$$P\mathcal{S}\mathcal{A}x = \frac{\langle \mathcal{S}\mathcal{A}x,f\rangle f}{\langle f,f\rangle} = \frac{\langle x,\mathcal{A}^*\mathcal{S}f\rangle f}{\langle f,f\rangle} = \mathcal{A}^*\mathcal{S}Px. \qquad (11)$$

It should be noted that $\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{A}^*) = (\frac{\langle f,f\rangle}{\langle \mathcal{S}f,f\rangle}\sigma_p(\mathcal{A}))^*$holds for all normal operators.

### *3.2 Connection to the product $\mathcal{S}$-numerical range*

The concept of product numerical range of a given operator has been greatly studied during the last few decades, ( see(Bakić, 1998)(Gawron, Piotr. "Z. Pucha la, JA Miszczak, L. Skowronek, K.

Zyczkowski., 2011)(Muiruri, 2018)(Zhang, D., L. Hou, and L. Ma., 2017)(Waed D., Joachim K. and Nazife E. Ö., 2018)and reference therein) and its usefulness in quantum theory has been defined. In particular, Marcus introduced the idea of decomposable numerical range (Wlat Hamad, Ahmed Muhammad, 2020) and reference therein. In this section we are going to define the definition of product $\mathcal{S}$-numerical range, of an operator $\mathcal{A}$, and An similar idea is established for operators working on a composite Hilbert space with a tensor product structure, which we investigate. Assume that the underlying Hilbert space $\mathcal{H}$ is given as a tensor product of two (separable) Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_l$ where $\mathcal{H}$ is finite-dimensional of composite dimension $n = kl$ where $dim\,\mathcal{H}_k = k$ and $dim\,\mathcal{H}_l = l$.

$$\mathcal{H} = \mathcal{H}_k \otimes \mathcal{H}_l. \qquad (12)$$

In this section we are going to define the following definitions.

**Definition 3.6** *Let $\mathcal{A}$ be a bounded linear operator and $\mathcal{S}$ be a self-adjoint operator both on a separable complex Hilbert space $\mathcal{H}_k \otimes \mathcal{H}_l$ the product $\mathcal{S}$-numerical range is defined as*

$$\Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) := \left\{ \begin{matrix} \dfrac{\langle f_k \otimes f_l, \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), f_k \otimes f_l\rangle} : \langle \mathcal{S}(f_k \otimes f_l), \\ (f_k \otimes f_l)\rangle \neq 0, f_k \in \mathcal{H}_k \; and \;\; f_l \in \mathcal{H}_l \end{matrix} \right\}. \qquad (13)$$

In order to identify Eq.(13) as a $\mathcal{S}$-numerical range with respect to a family of projections we introduce

$$\tilde{\mathcal{P}} := \left\{ \begin{matrix} P \in \mathbb{P} : \exists f_k \in \mathcal{H}_k,\, f_l \in \mathcal{H}_l \;\; such \;\; that \\ P = (f_k \otimes f_l) \cdot \dfrac{\langle f_k \otimes f_l, \cdot \rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle} \end{matrix} \right\}, \qquad (14)$$

where $\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle \neq 0$. In other words, for any $P \in \tilde{\mathcal{P}}$ one has

$$Ph = (f_k \otimes f_l) \cdot \frac{\langle f_k \otimes f_l, h\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

for some elements $f_k \in \mathcal{H}_k$ and $f_l \in \mathcal{H}_l$ and all $h \in \mathcal{H}$.

***Some properties of product $\mathcal{S}$-numerical range.***

We give some basic properties concerning product $\mathcal{S}$-numerical range.

**Proposition 3.7:**

1. For all $\mathcal{A}, \mathcal{B} \in \mathbb{M}_n$ then $\Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A} + \mathcal{B}) \subset \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) + \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{B})$.
2. For all $\mathcal{A} \in \mathbb{M}_n$ and $\alpha \in \mathbb{C}$, then
$$\Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A} + \alpha I) = \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) + \bar{\alpha},$$
and $$\Lambda_{\mathcal{S}}^{\otimes}(\alpha A) = \bar{\alpha}\Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}).$$
3. For all $\mathcal{A} \in \mathbb{M}_{mn}$, unitary $U_1 \in \mathbb{M}_m$ and $U_2 \in \mathbb{M}_n$, then $\Lambda_{\mathcal{S}}^{\otimes}\big((U_1 \otimes U_2)^* \mathcal{A}(U_1 \otimes U_2)\big) = \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A})$.
4. The product $\mathcal{S}$-numerical does not need to be convex, as seen in the following example.

*Proof.*

1. Let $\lambda \in \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A} + \mathcal{B})$ then by definition 3.6 there exist $f_k \in \mathcal{H}_k \; and \;\; f_l \in \mathcal{H}_l$ such that $\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle \neq 0$ then

$$\lambda = \frac{\langle (f_k \otimes f_l), \mathcal{S}(\mathcal{A} + B)(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$
$$= \frac{\langle (f_k \otimes f_l),\; \mathcal{S}\mathcal{A}(f_k \otimes f_l) + \mathcal{S}B(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$
$$= \frac{\langle (f_k \otimes f_l), \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle} + \frac{\langle (f_k \otimes f_l), \mathcal{S}B(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$
$$= \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) + \Lambda_{\mathcal{S}}^{\otimes}(B).$$

From this it follows that

$$\Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}+\mathcal{B}) \subset \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) + \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{B}).$$

2. Let $\lambda \in \Lambda_{\mathcal{S}}^{\otimes}(\alpha\mathcal{A})$ then by definition 3.6 there exist $f_k \in \mathcal{H}_k$ *and* $f_l \in \mathcal{H}_l$ such that $\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle \neq 0$ then by properties of inner product and linear operators we have

$$\lambda = \frac{\langle (f_k \otimes f_l), \mathcal{S}\alpha\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

$$= \frac{\bar{\alpha}\langle (f_k \otimes f_l), \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle} = \bar{\alpha}\Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A})$$

Moreover, for $\lambda \in \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}+\alpha I)$ from the first property and second, we can esaily see that

$$\lambda = \frac{\langle (f_k \otimes f_l), \mathcal{S}(\mathcal{A}+\alpha I)(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle} = \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) + \bar{\alpha} \quad .$$

Therefore $\Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}+\alpha I) \subset \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) + \bar{\alpha}$.

For the next direction, when $\lambda \in \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) + \bar{\alpha}$ then

$$\lambda = \frac{\langle (f_k \otimes f_l), \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle} + \bar{\alpha}$$

$$= \frac{\langle (f_k \otimes f_l), \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle} + \frac{\bar{\alpha}\langle (f_k \otimes f_l), \mathcal{S}I(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

$$= \frac{\langle (f_k \otimes f_l), \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle} + \frac{\langle (f_k \otimes f_l), \alpha\mathcal{S}I(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

$$= \frac{\langle (f_k \otimes f_l), \mathcal{S}(\mathcal{A}+\alpha)(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

Therefore $\lambda \in \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}) + \bar{\alpha}$ implies that

$$\lambda \in \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}+\alpha I).$$

3. Let $\lambda \in \Lambda_{\mathcal{S}}^{\otimes}((U_1 \otimes U_2)^*\mathcal{A}(U_1 \otimes U_2))$ for unitary operators $U_1 \in \mathbb{M}_m$ and $U_2 \in \mathbb{M}_n$ there exist $f_k \in \mathcal{H}_k$ *and* $f_l \in \mathcal{H}_l$ such that $\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle \neq 0$

$$\lambda = \frac{\langle (f_k \otimes f_l), \mathcal{S}\big((U_1 \otimes U_2)^*\mathcal{A}(U_1 \otimes U_2)\big)(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

$$= \frac{\langle (U_1 \otimes U_2)(f_k \otimes f_l), \mathcal{S}\mathcal{A}(U_1 \otimes U_2)(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

By properties of product and unitary operators we have

$$\lambda = \frac{\langle (U_1 f_k \otimes U_2 f_l), \mathcal{S}\mathcal{A}(U_1 f_k \otimes U_2 f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

$$= \frac{\langle (f_k \otimes f_l), \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), (f_k \otimes f_l)\rangle}$$

Therefore $\Lambda_{\mathcal{S}}^{\otimes}((U_1 \otimes U_2)^*\mathcal{A}(U_1 \otimes U_2)) \subseteq \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A})$. Similary we can see the next direction.

To proof 4. We give the following example.

**Example 3.1:** Let's investgate eigenvaluses of an operator $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$ and let $\mathcal{S}$ be $4 \times 4$ identity operator then we know that $D$ is normal operator with eigenvaluses $1, 0, 0$ and $i$. Therefore we observe that eigenvaluses 1 and $i$ are contains in $\Lambda_{\mathcal{S}}^{\otimes}(D)$, but $\frac{1+i}{2}$ are not.

We are going to establish the following results.

**Theorem 3.8** *Let $\mathcal{A}$ be a bounded linear operator and $\mathcal{S}$ be a self-adjoint operator both on a separable complex Hilbert space $\mathcal{H}_k \otimes \mathcal{H}_l$, then $W_{\mathcal{S},\tilde{\mathcal{P}}}(\mathcal{A}) = \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A})$.*

*Proof.* We choose an element $\lambda \in \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A})$ then there exist $f_k \in \mathcal{H}_k$ and $f_l \in \mathcal{H}_l$ , with

$$\lambda = \frac{\langle f_k \otimes f_l, \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), f_k \otimes f_l\rangle}$$

Now, we define $P$ to be projection onto the one-dimentional subspace spanned by $f_k \otimes f_l$ *it means* $P = (f_k \otimes f_l) \cdot \frac{\langle f_k \otimes f_l, \cdot\rangle}{\langle \mathcal{S}(f_k \otimes f_l), f_k \otimes f_l\rangle}$ . Therefore $P \in \tilde{\mathcal{P}}$. Addithionally,

$$(P\mathcal{S}\mathcal{A})(f_k \otimes f_l) = (f_k \otimes f_l) \cdot \frac{\langle f_k \otimes f_l, \mathcal{S}\mathcal{A}(f_k \otimes f_l)\rangle}{\langle \mathcal{S}(f_k \otimes f_l), f_k \otimes f_l\rangle} \qquad (15)$$

Thus $\lambda \in W_{\mathcal{S},\tilde{\mathcal{P}}}(\mathcal{A})$. Conversly: Suppose $\lambda \in W_{\mathcal{S},\tilde{\mathcal{P}}}(\mathcal{A})$ there exist vectors $g_k \in \mathcal{H}_k$ and

$g_l \in \mathcal{H}_l$ for which $(P\mathcal{S}\mathcal{A})(g_k \otimes g_l) = \lambda(g_k \otimes g_l)$, with $P = (g_k \otimes g_l) \cdot \frac{\langle g_k \otimes g_l, \cdot\rangle}{\langle \mathcal{S}(g_k \otimes g_l), g_k \otimes g_l\rangle}$.

Then $\lambda(g_k \otimes g_l) = (P\mathcal{S}\mathcal{A})(g_k \otimes g_l)$

$$= (g_k \otimes g_l) \cdot \frac{\langle g_k \otimes g_l, \mathcal{S}\mathcal{A}(g_k \otimes g_l)\rangle}{\langle \mathcal{S}(g_k \otimes g_l), g_k \otimes g_l\rangle}.$$

Hence it follows that

$$\lambda = \frac{\langle g_k \otimes g_l, \mathcal{S}A(g_k \otimes g_l)\rangle}{\langle \mathcal{S}(g_k \otimes g_l), g_k \otimes g_l\rangle} \in \Lambda_{\mathcal{S}}^{\otimes}(\mathcal{A}).$$

**Corollary 3.9** *Let $\mathcal{A}$ be a bounded linear operator and $\mathcal{S}$ be a self-adjoint operator both on a separable complex Hilbert space $\mathcal{H}_1 \otimes, \dots, \otimes \mathcal{H}_j$ we have*

$W_{\mathcal{S},\tilde{\mathcal{P}}(\mathcal{H}_1 \otimes \mathcal{H}_2, \otimes, \dots, \otimes \mathcal{H}_j)}(\mathcal{A}) = \Lambda^{\otimes}_{\mathcal{S},(\mathcal{H}_1 \otimes, \dots, \otimes \mathcal{H}_j)}(\mathcal{A})$.

*Proof.* First we consider $\lambda \in \Lambda^{\otimes}_{\mathcal{S},(\mathcal{H}_1 \otimes, \dots, \otimes \mathcal{H}_j)}(\mathcal{A})$ then there is $f_m \in \mathcal{H}_m$ for $m = 1,2, \dots, j$, with $\langle \mathcal{S}(f_1 \otimes f_2 \otimes, \dots, \otimes f_j), f_1 \otimes f_2 \otimes, \dots, \otimes f_j\rangle \neq 0$.

Then

$$\lambda = \frac{\langle f_1 \otimes f_2 \otimes, \dots, \otimes f_j, \mathcal{S}\mathcal{A}(f_1 \otimes f_2 \otimes, \dots, \otimes f_j)\rangle}{\langle \mathcal{S}(f_1 \otimes f_2 \otimes, \dots, \otimes f_j), f_1 \otimes f_2 \otimes, \dots, \otimes f_j\rangle}.$$

In this case, we define $P$ to be projection onto the subspace spanned by $f_1 \otimes f_2 \otimes, \dots, \otimes f_j$ it means

$P = (f_1 \otimes f_2 \otimes, \dots, \otimes f_j) \cdot \frac{\langle f_1 \otimes f_2 \otimes, \dots, \otimes f_j, \cdot\rangle}{\langle \mathcal{S}(f_1 \otimes f_2 \otimes, \dots, \otimes f_j), f_1 \otimes f_2 \otimes, \dots, \otimes f_j\rangle}$,

therefore $P \in \tilde{\mathcal{P}}$. So we have

$$(P\mathcal{S}\mathcal{A})(f_1 \otimes f_2 \otimes, \dots, \otimes f_j) = (f_1 \otimes f_2 \otimes, \dots, \otimes f_j) \cdot \frac{\langle f_1 \otimes f_2 \otimes, \dots, \otimes f_j, \mathcal{S}\mathcal{A}(f_1 \otimes f_2 \otimes, \dots, \otimes f_j)\rangle}{\langle \mathcal{S}(f_1 \otimes f_2 \otimes, \dots, \otimes f_j), f_1 \otimes f_2 \otimes, \dots, \otimes f_j\rangle}$$

$$= \lambda(f_1 \otimes f_2 \otimes, \dots, \otimes f_j). \qquad (16)$$

Hence $\lambda \in W_{\mathcal{S},\tilde{\mathcal{P}}(\mathcal{H}_1 \otimes, \dots, \otimes \mathcal{H}_j)}(\mathcal{A})$.

Conversely, assume that $\lambda \in W_{\mathcal{S},\tilde{\mathcal{P}}(\mathcal{H}_1 \otimes, \dots, \otimes \mathcal{H}_j)}(\mathcal{A})$again there exist vectors $g_m \in \mathcal{H}_m$ such that $\langle \mathcal{S}(g_1 \otimes g_2 \otimes, \dots, \otimes g_j), g_1 \otimes g_2 \otimes, \dots, \otimes g_j\rangle \neq 0$and

$(P\mathcal{S}\mathcal{A})(g_1 \otimes g_2 \otimes, \dots, \otimes, g_j) = \lambda(g_1 \otimes g_2 \otimes, \dots, \otimes, g_j)$.

We have

$P = (g_1 \otimes g_2 \otimes, \dots, \otimes, g_j) \cdot \frac{\langle g_1 \otimes g_2 \otimes, \dots, \otimes, g_j, \cdot\rangle}{\langle \mathcal{S}(g_1 \otimes g_2 \otimes, \dots, \otimes g_j), g_1 \otimes g_2 \otimes, \dots, \otimes g_j\rangle}$.

which implies that

$$(P\mathcal{S}\mathcal{A})(g_1 \otimes g_2 \otimes, \dots, \otimes, g_j) = (g_1 \otimes g_2 \otimes, \dots, \otimes, g_j) \cdot \frac{\langle g_1 \otimes g_2 \otimes, \dots, \otimes, g_j, \mathcal{S}\mathcal{A}(g_1 \otimes g_2 \otimes, \dots, \otimes g_j)\rangle}{\langle \mathcal{S}(g_1 \otimes g_2 \otimes, \dots, \otimes g_j), g_1 \otimes g_2 \otimes, \dots, \otimes g_j\rangle}.$$

Consequently,

$$\lambda = \frac{\langle g_1 \otimes g_2 \otimes, \dots, \otimes g_j, \mathcal{S}\mathcal{A}(g_1 \otimes g_2 \otimes, \dots, \otimes g_j)\rangle}{\langle \mathcal{S}(g_1 \otimes g_2 \otimes, \dots, \otimes g_j), g_1 \otimes g_2 \otimes, \dots, \otimes g_j\rangle}$$

and $\lambda \in \Lambda^{\otimes}_{c,(\mathcal{H}_1 \otimes, \dots, \otimes \mathcal{H}_j)}(\mathcal{A})$.

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