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A New Integral Transform and Applications: HK-Transform

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ABSTRACT

In this paper, we propose a new integral transform called the HK Transform and employ it to analytically solve the Volterra integral equations (VIE.) of the first kind. To achieve this First we derive the HK transform of basic algebraic and transcendental mathematical functions. The fundamental characteristics of the HK transform, which can be used to solve a variety of functional equations, including, we then discuss the fundamental properties, such as ordinary differential equations and integral equations, then we discover the exact solution for general first class V.I.E. To illustrate the applicability of the HK transform, numerical problems are examined and handled carefully step-by-step. The results show that the proposed HK Transform new integral transform produces accurate solutions for first-kind VIE. without requiring time-consuming computations.

1.Introduction

In the present situation, researchers have three key advantages over other mathematical methods when it comes to solving problems in science, social science, and engineering: simplicity, accuracy, and the ability to provide results without requiring laborious calculations. These attributes make integral transforms the first choice among other mathematical methods. Laplace(Joel L. Schiff, 1975), Several new integral transforms have been discovered recently, including the Kamal(Kamal Sedeeg, 2016), Sumudu(Kiliçman, Eltayeb and Agarwal, 2010), Sawi(Hilmi, MohammedFaeq and Fatah, 2024), Elzaki(Al-rikabi, 2022), Aboodh (Aboodh, 2013), Mohand(Aggarwal, Sharma and Gupta, 2019),Lplace(Joel L. Schiff, 1975; Hilmi and Jwamer, 2022) Sumudu(Kiliçman, Eltayeb and Agarwal, 2010), Sawi(Hilmi, MohammedFaeq and Fatah, 2024), Sadik (Latif Shaikh, 2018), Sawi [9],Rishi(Turab *et al.*, 2024)Anuj(Kumar, Bansal Shikha and Aggarwal, 2022) are only a few of the new integral transforms that researchers have discovered recently. Researchers have examined the well-known issues of growth and decay and solved modelling in mechanic , health science and biotechnology using a variety of integral transforms, such as the (Laplace (Murray R.Spiegel, 1965), Mohand (Aggarwal, Sharma and Gupta, 2019), Kushare(Patil, Shweta and Gatkal, 2022; Patil, Borse and Kapadi, 2023), Soham (Patil, Suryawanshi and Nehete, 2022)Sawi(Aggarwal and Singh, 2019). (Higazy and Aggarwal, 2021) found precise solutions to chemical kinetics problems by modeling with the Sawi transform using ordinary differential equations. El-Mesady et al(El-Mesady, Hamed and Alsharif, 2021). employed the Jafari transform to resolve a medical issue.

Many transformation can be used to solve ordinary differential equations, partial differential equations, integral equations (Attaweel, Almassry and Khyar, 2019; Al-rikabi, 2022; Kumar, Bansal Shikha and Aggarwal, 2022; Patil, Shweta and Gatkal, 2022; Patil, Suryawanshi and Nehete, 2022).

The goal of this study is to introduce the HK Transform, a new integral transform with basic features, and to use it to find the solutions to the first kind VIE. with a convolution type kernel, ordinary differential equations (ODEs), the exact solutions are obtained without large computational effort, The HK Transform, is superior to other known transforms because it gives precise solutions without requiring time-consuming calculation. The HK transform is twin to the widely used and well-known Laplace transform.

The paper systematically introduces the HK-Transform, beginning with Section 2 where its basic definition and initial mathematical properties are established.in section 3 linearity property has been showed, Section 4 applies the HK -Transform to specific functions, demonstrating its utility across algebraic, trigonometric, and exponential domains. Fundamental properties such as linearity, scaling, and potentially the convolution theorem is explored in Section 5, followed by Sections 6 and 7, which delve into differentiation and integration with respect to the HK-Transform.In section 8, we focus on the variable coefficient's theorem, Sections 9 and 10 focus on the inverse transform and further explore its linearity properties. Practical applications are showcased in Section 11, illustrating how the HK-Transform can effectively solve differential and integral equations. Finally, Section 12 consolidates these findings in a concluding summary, highlighting the transformative potential of the HK-Transform in mathematical applications and suggesting avenues for future research.

2. Definition of HK Transform

A function $u(t)$ is known original function such that:

1. $u(t) \equiv 0$ for $t < 0$,
2. $|u(t)| < Me^{s_0 t}$ for $t > 0$ with $M > 0, s_0 \in \mathbb{R}$.

We consider functions in the set A, defined by a novel transform known as the HK transform, defined for functions of exponential order

$$A = \left\{u(t): \exists\, N, s_1, s_2 > 0 \text{ such that } u(t) < Ne^{\frac{|t|}{s_j}}, \text{if } t \in [0,\infty)\right\}$$

For a given function in set A, the constant N must be finite number, while s_1, s_2 may be finite or infinite

The HK transform of an exponential order piecewise continuous function, denoted by the operator $HK(.)$, $u(t)$ and $s_1, s_2 > 0$,defined in the interval,$(0,\infty)$ is given by

$$HK\{u(t)\} = \frac{1}{\sqrt{\rho}}\int_0^\infty e^{-t\rho}u(t)dt = U(\rho), \quad t > 0 \quad , s_1 < \rho < s_2$$

In this transform, the variable ρ is employed to factor the variable t in the function $u(t)$'s argument. This transform is related to the Aboodh, Laplace, Elzaki, and Sadik transforms.

Remark: The HK transform has a duality relation with famous and mostly used integral transform "Laplace transform", if $L\big(f(t)\big) = \int_0^\infty f(t)e^{-st}dt = F(s),$ where L is Laplace transform, and $HK\big(u(t)\big) = U(\rho)$ where HK is HK transform , then $U(\rho) = \frac{F(\rho)}{\sqrt{\rho}}$.

Theorem 1: [**Sufficient Condition for Existence of the HK Transform**]: The HK transform $HK\{u(t)\}$ exists for functions of exponential order if $\int_0^b |u(t)|dt$ exists for $b > 0$.

Proof:

$$\frac{1}{\sqrt{\rho}}\int_0^\infty |u(t)e^{-t\rho}|dt = \frac{1}{\sqrt{\rho}}\int_0^n |u(t)e^{-t\rho}|dt + \frac{1}{\sqrt{\rho}}\int_n^\infty |u(t)e^{-t\rho}|dt,$$ where $n \in (0,\infty)$

Since $t > 0 \Longrightarrow -t < 0 \Longrightarrow -t\rho < 0 \Longrightarrow e^{-t\rho} < 1$ so we get

$$\le \frac{1}{\sqrt{\rho}}\int_0^n |u(t)|dt + \frac{1}{\sqrt{\rho}}\int_0^\infty |u(t)|e^{-t\rho}dt,$$

$$\le \frac{1}{\sqrt{\rho}}\int_0^n |u(t)|dt + M\frac{1}{\sqrt{\rho}}\int_0^\infty e^{\alpha t}e^{-t\rho}dt,$$

$$= \frac{1}{\sqrt{\rho}}\int_0^n |u(t)|dt + M\frac{1}{\sqrt{\rho}}\int_0^\infty e^{-(\rho-\alpha)t}dt,$$

$$= \frac{1}{\sqrt{\rho}}\int_0^n |u(t)|dt + \frac{M}{-\sqrt{\rho}(\rho-\alpha)}\, lim_{s\to\infty}\, e^{-(\rho-\alpha)t}\,|_0^s,$$

$$= \frac{1}{\sqrt{\rho}}\int_0^n |u(t)|dt + \frac{M}{\sqrt{\rho}(\rho-\alpha)}.$$

The first integral $\frac{1}{\sqrt{\rho}}\int_0^n |u(t)|dt$ exists, and the second term $\frac{M}{\sqrt{\rho}(\rho-\alpha)}$ is finite for $\rho > \alpha$ so the integral $\frac{1}{\sqrt{\rho}}\int_0^\infty u(t)e^{-t\rho}dt$ converges absolutely and the HK transform $HK\{u(t)\}$ exists.

3. Linearity property of HK transform

Theorem 2: If $HK\{u(t)\} = U(\rho)$, then $HK\{\sum_{i=1}^n \alpha_i u_i(t)\} = \alpha_i \sum_{i=1}^n U_i(\rho)$, where α_i are arbitrary constant.

Proof: From the definition of HK transform we have $HK\{u(t)\} = \frac{1}{\sqrt{\rho}}\int_0^\infty e^{-t\rho}u(t)dt = U(\rho),$

$$\Rightarrow HK\{\textstyle\sum_{i=1}^n \alpha_i u_i(t)\} = \frac{1}{\sqrt{\rho}}\int_0^\infty \sum_{i=1}^n \alpha_i u_i(t)e^{-t\rho}dt,$$

$$\Rightarrow HK\{\textstyle\sum_{i=1}^n \alpha_i u_i(t)\} = \sum_{i=1}^n \alpha_i \int_0^\infty \frac{1}{\sqrt{\rho}}e^{-t\rho}u_i(t)dt,$$

$$\Rightarrow HK\{\textstyle\sum_{i=1}^n \alpha_i u_i(t)\} = \sum_{i=1}^n \alpha_i\, U_i(\rho).$$

4. HK transform for some functions

In this section we derive the HK Transform for most useable functions.

Type 1: Consider the constant function $u(t) = c$, where $c \in R$, From the definition of HK transform, we have:

$$HK\{c\} = \frac{1}{\sqrt{\rho}}\int_0^\infty e^{-t\rho}c\,dt,$$

$$HK\{c\} = \frac{c}{\sqrt{\rho}}\int_0^\infty e^{-t\rho}dt = \frac{c}{\sqrt{\rho}}\left\{\frac{-1}{\rho}e^{-t\rho}\Big|_0^\infty\right\},$$

So $HK\{c\} = \frac{c}{\rho\sqrt{\rho}} = \frac{c}{\rho^{\frac{3}{2}}}$.

Type 2: Consider the function $u(t) = t$,

From the definition of HK transform, we have:

$$HK\{t\} = \frac{1}{\sqrt{\rho}}\int_0^\infty te^{-t\rho}dt,$$

Using integrating by part

$$HK\{t\} = \frac{1}{\sqrt{\rho}}\left\{\frac{-t}{\rho}e^{-t\rho}\Big|_0^\infty + \int_0^\infty \frac{1}{\rho}e^{-t\rho}dt\right\},$$

$$HK\{t\} = \frac{1}{\sqrt{\rho}}\left\{-\frac{1}{\rho^2}e^{-t\rho}\right\}_0^{\infty} = \frac{1}{\rho^{\rho}\sqrt{\rho}} = \frac{1}{\rho^{\frac{5}{2}}}.$$

Type 3: Consider the function $u(t) = t^2$.
From the definition of HK transform, we have:

$$HK\{t^2\} = \frac{1}{\sqrt{\rho}}\int_0^{\infty} t^2 e^{-t\rho}dt,$$

$$HK\{t^2\} = \frac{-1}{\sqrt{\rho}}\left\{e^{-t\rho}\left\{\frac{t^2}{\rho} + \frac{2t}{\rho^2} + \frac{2}{\rho^3}\right\}\right\}_0^{\infty},$$

$$= \frac{2!}{\rho^3\sqrt{\rho}} = \frac{2!}{\rho^{7/2}},$$

Type 4: if we consider the function $u(t) = t^n$, then the HK transform becomes: $HK\{t^n\} = \frac{n!}{\rho^{n+\frac{3}{2}}}$

Mathematical induction can be used to prove this result.

Type 5: Consider the function $u(t) = t^{\beta}; \quad t > 0; \quad \beta > -1$, From the definition of HK transform, we have:

$$HK\{t^{\beta}\} = \frac{1}{\sqrt{\rho}}\int_0^{\infty} t^{\beta}e^{-t\rho}dt,$$ using integrate by part we get

$$HK\{t^{\beta}\} = \frac{1}{\sqrt{\rho}}\left\{\frac{-t^{\beta}}{\rho}e^{-t\rho}\Big|_0^{\infty} + \frac{\beta}{\rho}\int_0^{\infty} t^{\beta-1}e^{-t\rho}dt\right\},$$

$$HK\{t^{\beta}\} = \frac{1}{\sqrt{\rho}}\left\{\frac{\beta}{\rho}\cdot\frac{\Gamma(\beta)}{\rho^{\beta}}\right\} = \frac{\Gamma(\beta+1)}{\rho^{\beta+3/2}}.$$

Type 6: Consider the function $u(t) = e^{at}$
From the definition of HK transform, we have:

$$HK\{e^{at}\} = \frac{1}{\sqrt{\rho}}\int_0^{\infty} e^{-t\rho}e^{at}dt = \frac{1}{\sqrt{\rho}}\int_0^{\infty} e^{-t(\rho-a)}dt,$$

$$HK\{e^{at}\} = \frac{1}{\sqrt{\rho}(\rho-a)} = \frac{1}{\rho^{\frac{3}{2}}-a\rho^{\frac{1}{2}}}.$$

Type 7: Consider the function $u(t) = sin(at)$.
From the definition of HK transform, we have: Using Euler's formula, $sin(at)$ can be expressed as: $sin(at) = \frac{e^{ait}-e^{-ait}}{2i}$

$$HK\{sin(at)\} = HK\left\{\frac{e^{ait}-e^{-ait}}{2i}\right\},$$

$$= \frac{1}{2i}\{HK(e^{ait} - e^{-ait})\},$$ using the result from type 6

$$= \frac{1}{2i\sqrt{\rho}}\left\{\frac{1}{\rho-ai} - \frac{1}{\rho+a_i}\right\},$$

So $HK\{sin(at)\} = \frac{1}{2i\sqrt{\rho}}\left\{\frac{\rho+a_i-\rho+a_i}{(\rho-a_i)(\rho+a_i)}\right\},$

$$HK\{sin(at)\} = \frac{a}{\sqrt{\rho}(\rho^2+a^2)}.$$

Type 8: Consider the function $u(t) = cos(at)$
From the definition of HK transform, we have: Using Euler's formula, $cos(at)$ can be expressed as: $cos(at) = \frac{e^{ait}+e^{-ait}}{2}$, so

$$HK\{cos(at)\} = HK\left\{\frac{e^{ait}+e^{-ait}}{2}\right\},$$

$$HK\{cos(at)\} = \frac{1}{2}\{HK(e^{ait}) + HK(e^{-ait})\},$$

using the result from type 6

$$HK\{cos(at)\} = \frac{1}{2\sqrt{\rho}}\left\{\frac{1}{\rho-ai} + \frac{1}{\rho+ai}\right\},$$

$$HK\{cos(at)\} = \frac{\sqrt{\rho}}{\rho^2+a^2},$$

Type 9: Consider the function $u(t) = sinh(at)$
From the definition of HK transform, we have: Using the definition of the hyperbolic Sine function: $sinh(at) = \frac{e^{at}-e^{-at}}{2}$

$$HK\{sinh(at)\} = HK\left\{\frac{e^{at}-e^{-at}}{2}\right\},$$

$$HK\{sinh(at)\} = \frac{1}{2}\{\{HK\{e^{at}\} - HK\{e^{-at}\}\},$$ using the result from type 6

$$HK\{sinh(at)\} = \frac{1}{2\sqrt{\rho}}\left\{\frac{1}{\rho-a} - \frac{1}{\rho+a}\right\},$$

$$HK\{sinh(at)\} = \frac{a}{\sqrt{\rho}(\rho^2-a^2)}.$$

Type 10: Consider the function $u(t) = cosh(at)$
From the definition of HK transform, we have: Using the definition of the hyperbolic cosine function: $cosh(at) = \frac{e^{at}+e^{-at}}{2}$

$$HK\{cosh(at)\} = HK\left\{\frac{e^{at}+e^{-at}}{2}\right\},$$

$$HK\{cosh(at)\} = \frac{1}{2}\{HK(e^{at}) + HK(e^{-at})\},$$ using the result from type 6

$$= \frac{1}{2\sqrt{\rho}}\left\{\frac{1}{\rho-a} + \frac{1}{\rho+a}\right\},$$

$$= \frac{\sqrt{\rho}}{\rho^2-a^2}.$$

5. Fundamental properties of HK transform

5.1. Scaling property of HK transform

Theorem 3: if $HK\{u(t)\} = U(\rho)$, then $HK\{u(kt)\} = \frac{1}{k\sqrt{k}}U\left(\frac{\rho}{k}\right)$.

Proof: from the definition of HK transform

$$HK\{u(kt)\} = \frac{1}{\sqrt{\rho}}\int_0^{\infty} e^{-t\rho}u(kt)dt,$$

Let $s = kt \Rightarrow ds = kdt$

$$HK\{u(kt)\} = \frac{1}{k\sqrt{\rho}}\int_0^{\infty} e^{-\frac{s}{k}\rho}u(s)ds,$$

$$HK\{u(kt)\} = \frac{1}{k\sqrt{k}\sqrt{\frac{\rho}{k}}}\int_0^\infty e^{-s\left(\frac{\rho}{k}\right)}u(s)ds,$$
$$HK\{u(kt)\} = \frac{1}{k\sqrt{k}}U\left(\frac{\rho}{k}\right).$$

5.2. Shifting property of HK transform

Theorem 4: If $HK\{u(t)\} = U(\rho)$, then $HK\{e^{at}u(t)\} = \sqrt{1-\frac{a}{\rho}}U(\rho-a)$.

Proof: let $HK\{u(t)\} = U(\rho)$,
from the definition of HK transform
$$HK\{e^{at}u(t)\} = \frac{1}{\sqrt{\rho}}\int_0^\infty e^{-t(\rho-a)}u(t)dt,$$
$$= \frac{\sqrt{\rho-a}}{\sqrt{\rho}}\cdot\frac{1}{\sqrt{\rho-a}}\int_0^\infty e^{-t(\rho-a)}u(t)dt,$$
$$= \sqrt{1-\frac{a}{\rho}}U(\rho-a).$$

5.3. (Convolution) property of HK transform

Theorem 5: If $HK\{u(t)\} = U(\rho)$ and $HK\{g(t)\} = G(\rho)$, then $HK\{u(t)*g(t) = \sqrt{\rho}U(\rho)G(\rho)$ where the convolution of $u(t)$and $g(t)$ denoted by $u(t)*g(t) = \int_0^t u(t-s)g(s)ds = \int_0^t u(s)g(t-s)ds$.

proof: From the definition of convolution and HK transform, we have
$$HK\{u(t)*g(t)\} = \frac{1}{\sqrt{\rho}}\int_0^\infty e^{-t\rho}\int_0^t u(s)g(t-s)dsdt$$
$$= \frac{1}{\sqrt{\rho}}\int_0^\infty u(s)\int_s^\infty g(t-s)e^{-t\rho}ds\Big)dt,$$
Let $v = t-s \Rightarrow dv = dt$,
$$HK\{u(t)*g(t)\} = \frac{1}{\sqrt{\rho}}\int_0^\infty u(s)\int_0^\infty g(v)e^{-\rho(s+v)}dsdv,$$
$$= \frac{1}{\sqrt{\rho}}\int_0^\infty u(s)e^{-s\rho}ds\cdot\int_0^\infty g(v)e^{-v\rho}dv,$$
$$= \sqrt{\rho}*\frac{1}{\sqrt{\rho}}\left[\int_0^\infty e^{-s\rho}u(s)ds\right]*\frac{1}{\sqrt{\rho}}\left[\int_0^\infty e^{-v\rho}g(v)dv\right],$$
$$= \sqrt{\rho}\,U(\rho)\,G(\rho).$$

6. HK Transform of Derivatives

Theorem 6: Let $HK\{u(t)\} = U(\rho)$, then

a) $HK(u'(t)) = \rho U(\rho) - \rho^{-\frac{1}{2}}u(0)$,

b) $HK\{u''(t)\} = \rho^2 u(\rho) - \rho^{\frac{1}{2}}u(0) - \rho^{-\frac{1}{2}}u'(0)$,

c) $HK\{u'''(t)\} = \rho^3 U(\rho) - \rho^{\frac{3}{2}}u(0) - \rho^{\frac{1}{2}}u'(0) - \rho^{-\frac{1}{2}}u''(0)$,

d) $HK\{u^{(n)}(t)\} = \rho^n U(\rho) - \sum_{k=0}^{n-1}\rho^{k-\frac{1}{2}}u^{(n-1-k)}(0)$.

proof:

a) $HK\{u'(t)\} = \frac{1}{\sqrt{\rho}}\int_0^\infty e^{-t\rho}u'(t)dt$

let $u = e^{-t\rho} \Rightarrow du = -\rho e^{-t\rho}dt$, and $dv = u'(t)dt \Rightarrow v = u(t)$
$$HK\{u'(t)\} = \frac{1}{\sqrt{\rho}}\left\{e^{-t\rho}u(t)|_0^\infty + \rho\int_0^\infty e^{-t\rho}u(t)dt\right\},$$
$$HK\{u'(t)\} = \frac{-1}{\sqrt{\rho}}u(0) + \frac{\rho}{\sqrt{\rho}}\int_0^\infty e^{-t\rho}u(t)dt,$$
$$HK\{u'(t)\} = \rho U(\rho) - \frac{1}{\sqrt{\rho}}u(0).$$

b) $HK\{u''(t)\} = HK\{(u'(t))'\}$
$$HK\{u''(t)\} = \rho HK\{u'(t)\} - \rho^{-\frac{1}{2}}u'(0),$$
$$HK\{u''(t)\} = \rho\left(\rho U(\rho) - \rho^{-\frac{1}{2}}u(0)\right) - \rho^{-\frac{1}{2}}u'(0),$$
$$HK\{u''(t)\} = \rho^2 U(\rho) - \rho^{\frac{1}{2}}u(0) - \rho^{-\frac{1}{2}}u'(0).$$

c) $HK\{u'''(t)\} = \rho HK\{u''(t)\} - \rho^{-\frac{1}{2}}u''(0)$
$$HK\{u'''(t)\} = \rho\left\{\rho^2 U(\rho) - \rho^{\frac{1}{2}}u(0) - \rho^{-\frac{1}{2}}u'(0)\right\} - \rho^{-\frac{1}{2}}u''(0)$$
$$HK\{u'''(t)\} = \rho^3 U(\rho) - \rho^{\frac{3}{2}}u(0) - \rho^{\frac{1}{2}}u'(0) - \rho^{-\frac{1}{2}}u''(0).$$

d) it is easy by mathematical induction.

7. HK Transform of integral of a function

Theorem 7: Let $HK\{u(t)\} = U(\rho)$, then $HK\left\{\int_0^t u(s)ds\right\} = \frac{U(\rho)}{\rho}$.

Proof: let $G(t) = \int_0^t u(s)ds$

Then $G'(t) = u(t)$, and $HK\{G'(t)\} = HK(u(t))$,

Thus, $\rho HK\{G(t)\} - \frac{1}{\sqrt{\rho}}G(0) = U(\rho)$,

$\rho HK\{G(t)\} = U(\rho) \rightarrow HK\{G(t)\} = \frac{U(\rho)}{\rho} \rightarrow HK\left\{\int_0^t u(s)ds\right\} = \frac{U(\rho)}{\rho}$.

8. HK-Transform of variable coefficients

Theorem 8: Let $HK\{u(t)\} = U(\rho),$ then:

a) $HK\{tu(t)\} = -\frac{d}{d\rho}\left(U(\rho)\right) - \frac{1}{2\rho}U(\rho)$.

b) $HK\{t^2u(t)\} = \frac{d^2}{d\rho^2}\left(U(\rho)\right) + \frac{1}{\rho}\frac{d}{d\rho}U(\rho) - \frac{1}{4\rho^2}U(\rho)$.

c) $HK\{tu'(t)\} = -\rho\frac{d}{d\rho}\left(U(\rho)\right) - \frac{3}{2}U(\rho)$.

Proof:

a) $HK\{u(t)\} = \frac{1}{\sqrt{\rho}}\int_0^\infty u(t)e^{-t\rho}dt = U(\rho)$

Now by fundamental integral theorem in general form we have

$\frac{d}{d\rho}\left(U(\rho)\right) =$
$\frac{1}{\sqrt{\rho}}\int_0^\infty -tu(t)e^{-t\rho}dt - \frac{1}{2\rho\sqrt{\rho}}\int_0^\infty u(t)e^{-t\rho}dt$
$\frac{d}{d\rho}\left(U(\rho)\right) = -HK\{tu(t)\} - \frac{1}{2\rho}U(\rho)$
$HK\{tu(t)\} = -\frac{d}{d\rho}\left(U(\rho)\right) - \frac{1}{2\rho}U(\rho).$

b) Similarly, for $t^2u(t)$: we have

$\frac{d^2}{d\rho^2}\left(U(\rho)\right) =$
$\frac{1}{\sqrt{\rho}}\int_0^\infty t^2u(t)e^{-t\rho}dt + \frac{1}{2\rho\sqrt{\rho}}\int_0^\infty tu(t)e^{-t\rho}dt +$
$\frac{1}{2\rho\sqrt{\rho}}\int_0^\infty tu(t)e^{-t\rho}dt + \frac{3}{4\rho^2\sqrt{\rho}}\int_0^\infty u(t)e^{-t\rho}dt,$
$\frac{d^2}{d\rho^2}\left(U(\rho)\right) = HK\{t^2u(t)\} + \frac{3}{4\rho^2}U(\rho) +$
$\frac{1}{\rho}HK\{tu(t)\}$
$HK\{t^2u(t)\} = \frac{d^2}{d\rho^2}\left(U(\rho)\right) - \frac{3}{4\rho^2}U(\rho) -$
$\frac{1}{\rho}\left\{-\frac{d}{d\rho}\left(U(\rho)\right) - \frac{1}{2\rho}U(\rho)\right\}$
$HK\{t^2u(t)\} = \frac{d^2}{d\rho^2}\left(U(\rho)\right) + \frac{1}{\rho}\frac{d}{d\rho}\left(U(\rho)\right) - \frac{1}{4\rho^2}U(\rho).$

c) From part (**a**) we have $HK\{tu(t)\} =$
$-\frac{d}{d\rho}\left(U(\rho)\right) - \frac{1}{2\rho}U(\rho).$
And since $HK\{u(t)\} = U(\rho)$, Theat means
$HK\{tu(t)\} = -\frac{d}{d\rho}(HK\{u(t)\}) - \frac{1}{2\rho}HK\{u(t)\}$
Now replace $u(t)$ by $u'(t)$ we get $HK\{tu'(t)\} =$
$-\frac{d}{d\rho}(HK\{u'(t)\}) - \frac{1}{2\rho}HK\{u'(t)\}$
And from theorem 6 part (a) we have
$HK(u'(t)) = \rho U(\rho) - \rho^{-\frac{1}{2}}u(0)$
So $HK\{tu'(t)\} = -\frac{d}{d\rho}\left(\rho U(\rho) - \rho^{-\frac{1}{2}}u(0)\right) -$
$\frac{1}{2\rho}\left\{\rho U(\rho) - \rho^{-\frac{1}{2}}u(0)\right\},$
$HK\{tu'(t)\} = -\frac{d}{d\rho}\left(\rho U(\rho)\right) + \frac{d}{d\rho}\left(\rho^{-\frac{1}{2}}u(0)\right) -$
$\frac{1}{2}U(\rho) + \frac{1}{2}\rho^{-\frac{3}{2}}u(0),$
$HK\{tu'(t)\} = -\rho\frac{d}{d\rho}\left(U(\rho)\right) - U(\rho) - \frac{1}{2}\rho^{-\frac{3}{2}}u(0) -$
$\frac{1}{2}U(\rho) + \frac{1}{2}\rho^{-\frac{3}{2}}u(0),$
$HK\{tu'(t)\} = -\rho\frac{d}{d\rho}\left(U(\rho)\right) - \frac{3}{2}U(\rho).$

9. The Inverse HK Transform

If $HK\{u(t)\} = U(\rho)$, then the inverse HK transform is defined as $u(t) = HK^{-1}\{U(\rho)\}$, The corresponding inverse HK transform is:

$$u(t) = \frac{1}{2\pi i}\lim_{t\to\infty}\int_{\gamma-it}^{\gamma+it}\sqrt{\rho}U(\rho)e^{st}dt = L^{-1}[U(\rho)],$$

where $i = \sqrt{-1}$ and $\gamma \in \mathbb{R}$, in order for the convergence zone to contain the contour path of integration.

10. Linearity Property of Inverse HK Transform

Theorem 9: If
$HK\{u_i(t)\} = U_i(\rho)\ and\ HK^{-1}\{U_i(\rho)\} = u_i(t)$, then the inverse HK transform is also a linear operator $HK^{-1}\{\sum_{i=1}^{n}\alpha_iU_i(\rho)\} = \sum_{i=1}^{n}\alpha_iHK^{-1}\{U_i(\rho)\}$, where α_i are arbitrary constants.

11. Illustrative examples applications

This section lists six problems that demonstrate the utility of the HK transformation, these examples show how to find precise solutions for ordinary differential equations and Volterra integral equations of the first kind.

Problem 1:(R.kumar, J.Chandel and S.Aggarwal, 2022) solve the following integral equation $t + t^2 = \int_0^t cos(t-s)\,u(s)ds.$

Solution: The given integral equation is a Volterra integral equation of the first kind
$t + t^2 = \int_0^t cos(t-s)\,u(s)ds$
By taking HK transform operator for both sides we get
$HK\{t+t^2\} = HK\left\{\int_0^t cos(t-s)\,u(s)ds\right\}$ Using the convolution property (Theorem 5), we obtain:
$\frac{1}{\rho^{3/2}} + \frac{1}{\rho^{5/2}} = \sqrt{\rho}HK\{cos(t)\}HK\{u(t)\}$
$\frac{1}{\rho^{3/2}} + \frac{1}{\rho^{5/2}} = \sqrt{\rho}\frac{\sqrt{\rho}}{\rho^2+1}U(\rho)$
$\left(\frac{1}{\rho^{5/2}} + \frac{1}{\rho^{7/2}}\right)(\rho^2+1) = U(\rho)$
$U(\rho) = \frac{1}{\rho^{1/2}} + \frac{1}{\rho^{3/2}} + \frac{1}{\rho^{5/2}} + \frac{1}{\rho^{7/2}}$ by take inverse HK transform we get
$HK^{-1}\{U(\rho)\} = HK^{-1}\left\{\frac{1}{\rho^{1/2}} + \frac{1}{\rho^{3/2}} + \frac{1}{\rho^{5/2}} + \frac{1}{\rho^{7/2}}\right\}$
so, the exact solution is $u(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3}.$

Problem 2:(R.kumar, J.Chandel and S.Aggarwal, 2022) Consider the first kind VIE. with difference kernel defined as $sint = \int_0^t e^{t-s}u(s)ds.$

Solution: The I.E is $sin(t) = \int_0^t e^{t-s}u(s)ds$
Operating the HK transform for both sides, we get $HK\{sin(t)\} = HK\left\{\int_0^t e^{t-s}u(s)ds\right\}$
Using the convolution property (Theorem 5): $HK\{sin(t)\} = \frac{1}{\sqrt{\rho}(\rho^2+1)}$, and $HK\{e^{t-s}\} = \frac{1}{\sqrt{\rho}(\rho-1)}$, then

$\frac{1}{\sqrt{\rho}(\rho^2+1)} = \sqrt{\rho}\, HK\{e^{t-s}\}HK\{u(t)\}$,

$\frac{1}{\sqrt{\rho}(\rho^2+1)} = \sqrt{\rho}\, \frac{1}{\sqrt{\rho}(\rho-1)}U(\rho) \to U(\rho) = \frac{(\rho-1)}{\sqrt{\rho}(\rho^2+1)}$,

Take inverse HK transform for both sides we obtain the exact solution

$HK^{-1}\{U(\rho)\} = HK^{-1}\left\{\frac{\rho-1}{\sqrt{\rho}(\rho^2+1)}\right\} = HK^{-1}\left\{\frac{\rho}{\sqrt{\rho}(\rho^2+1)}\right\} - HK^{-1}\left\{\frac{1}{\sqrt{\rho}(\rho^2+1)}\right\}$,

$u(t) = cost - sint$.

Problem 3:(R.kumar, J.Chandel and S.Aggarwal, 2022) Consider the first kind integral equation $t = \int_0^t e^{t-s}u(s)ds$

Solution: The given integral equation is Volterra integral equation of first kind

$t = \int_0^t e^{t-s}u(s)ds$

By taking HK transform operator for both sides we get

$HK\{t\} = HK\left\{\int_0^t e^{t-s}u(s)ds\right\}$ from the convolution property in theorem 5 we obtain

$\frac{1}{\rho^{3/2}} = \sqrt{\rho}HK\{e^{t-s}\}HK\{u(t)\}$

$\frac{1}{\rho^{3/2}} = \sqrt{\rho}\frac{1}{\sqrt{\rho}(\rho-1)}U(\rho)$

$U(\rho) = \frac{1}{\rho^{1/2}} - \frac{1}{\rho^{3/2}}$ by take inverse HK transform we get $HK^{-1}\{U(\rho)\} = HK^{-1}\left\{\frac{1}{\rho^{1/2}} - \frac{1}{\rho^{3/2}}\right\}$

so, the exact solution is $u(t) = 1 - t$.

Remark: The above three problems were solved in (R.kumar, J.Chandel and S.Aggarwal, 2022) with a large computational effort. We obtained the same exact solutions more easily using the HK transform.

Problem 4:(Kharrat and Toma, 2020) Consider the first order differential equation

$u'' - 2u' - 3u = 0; u(0) = 1, u'(0) = 2$

Solution: Solve the differential equation using the HK transform:

Apply HK transform for both sides we get $HK\{u'' - 2u' - 3u\} = HK\{0\}$, Applying the HK transform and using the linearity property (Theorem 2) and the property for derivatives (Theorem 6), we get:

$\rho^2U(\rho) - \sqrt{\rho}u(0) - \frac{1}{\sqrt{\rho}}u'(0) - 2\rho U(\rho) + \frac{2}{\sqrt{\rho}}u(0) - 3U(\rho) = 0$

$(\rho^2 - 2\rho - 3)U(\rho) = \sqrt{\rho} \to U(\rho) = \frac{\rho}{\sqrt{\rho}(\rho^2-2\rho-3)} = \frac{\rho}{\sqrt{\rho}(\rho-3)(\rho+1)} = \frac{3}{4}\frac{1}{\sqrt{\rho}(\rho-3)} + \frac{1}{4}\frac{1}{\sqrt{\rho}(\rho+1)}$

By applying inverse HK transform for both sides, we get

$HK^{-1}\{U(\rho)\} = HK^{-1}\left\{\frac{3}{4}\frac{1}{\sqrt{\rho}(\rho-3)} + \frac{1}{4}\frac{1}{\sqrt{\rho}(\rho+1)}\right\}$. The exact solution is $u(t) = \frac{3}{4}e^{3t} + \frac{1}{4}e^{-t}$.

Problem 5: Consider the first order differential equation $u' - 2u = 0, \quad u(0) = 1$

Solution: Solve the differential equation by HK transform

Apply HK transform for both sides we get $HK\{u' - 2u\} = HK\{0\}$, from property of linearity in theorem 2 and from property HK transform for derivative in theorem 6 we have $HK\{u'\} = \rho U(\rho) - \frac{1}{\sqrt{\rho}}u(0)$, thus

$\rho U(\rho) - \frac{1}{\sqrt{\rho}}u(0) - 2U(\rho) = 0$, from initial condition we have $u(0) = 1$,

$(\rho - 2)U(\rho) = \frac{1}{\sqrt{\rho}} \to U(\rho) = \frac{1}{\sqrt{\rho}(\rho-2)}$ apply inverse HK transform for both sides we get

$HK^{-1}\{U(\rho)\} = HK^{-1}\left\{\frac{1}{\sqrt{\rho}(\rho-2)}\right\}$ the exact solution is $u(t) = e^{2t}$.

Problem 6:(Kharrat and Toma, 2020) Consider the first order differential equation $u'(t) + u(t) = 3, \quad u(0) = 1$

Solution: Solve the differential equation by HK transform

Apply HK transform for both sides we get $HK\{u' + u\} = HK\{3\}$ from property of linearity in theorem 2 and from property HK transform for derivative in theorem 6 we have $HK\{u'\} = \rho U(\rho) - \frac{1}{\sqrt{\rho}}u(0)$, thus

$\rho U(\rho) - \frac{1}{\sqrt{\rho}}u(0) + U(\rho) = \frac{3}{\rho\sqrt{\rho}}$, from initial condition we have $u(0) = 1$,

$(\rho + 1)U(\rho) = \frac{3}{\rho\sqrt{\rho}} + \frac{1}{\sqrt{\rho}} \to U(\rho) = \frac{3}{\rho\sqrt{\rho}(\rho+1)} +$

$\frac{1}{\sqrt{\rho}(\rho+1)}$

apply inverse HK transform for both sides we get

$HK^{-1}\{U(\rho)\} = HK^{-1}\left\{\frac{3}{\rho\sqrt{\rho}(\rho+1)} + \frac{1}{\sqrt{\rho}(\rho+1)}\right\}$,

$HK^{-1}\{U(\rho)\} = HK^{-1}\left\{\frac{3}{\rho\sqrt{\rho}} - \frac{3}{\sqrt{\rho}(\rho+1)} + \frac{1}{\sqrt{\rho}(\rho+1)}\right\} \Longrightarrow u(t) = HK^{-1}\left\{\frac{3}{\rho\sqrt{\rho}} - \frac{2}{\sqrt{\rho}(\rho+1)}\right\}$

So, the exact solution is $u(t) = 3 - 2e^{-t}$.

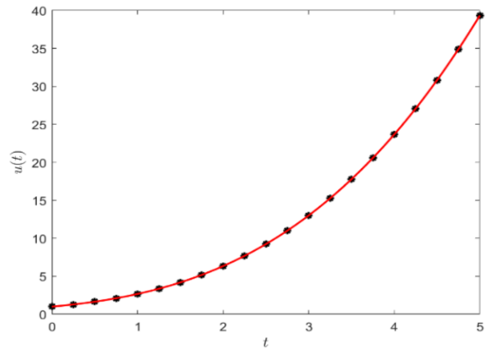

Figure.1: Exact solution for problem 1, where $t\epsilon[0,5]$

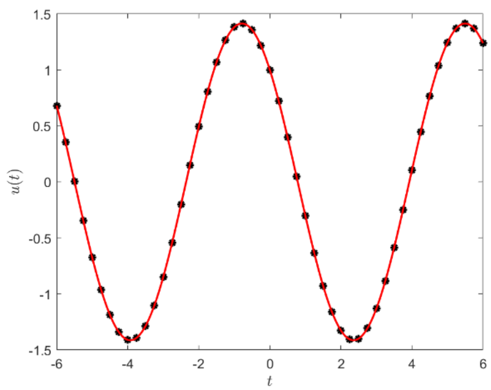

Figure.2: Exact solution for problem 2, where $t\epsilon[-2\pi, 2\pi]$

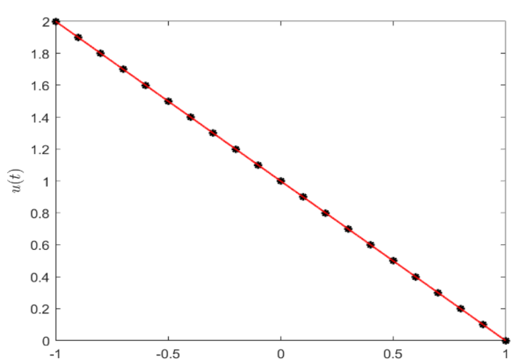

Figure.3: Exact solution for problem 3, where $t\epsilon[-1,1]$

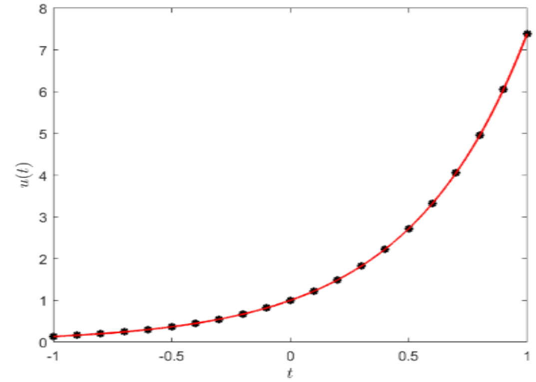

Figure.4: Exact solution for problem 4, where $t\epsilon[-1,1]$

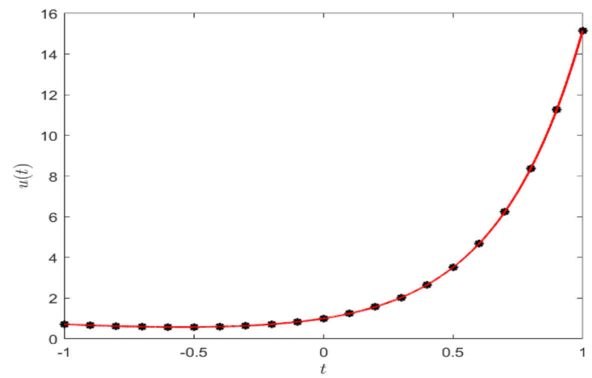

Figure.5: Exact solution for problem 5, where

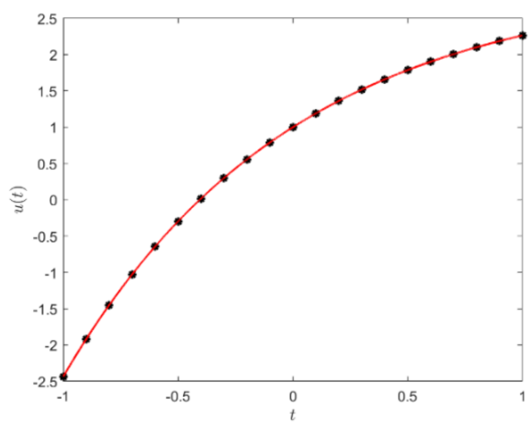

Figure.6: Exact solution for problem 6, where $t\epsilon[-1,1]$

Figure 1 displays the curve solution for Problem 1, with the time interval $t = [0, 5]$. It is evident from the graph that as the space variable increases, the function $u(t)$ also increases, suggesting a positive correlation between the space variable and $u(t)$. **Figure 2** illustrates the curve solution for Problem 2, over the time interval $t = [-2\pi, 2\pi]$. The periodic nature of $u(t)$ is apparent, indicating that the solution repeats itself at regular intervals, characteristic of periodic functions. **Figure 3** shows the curve solution for Problem 3, within the time interval $t = [-1,1]$. Here, it is clear that as the space variable increases, $u(t)$ decreases, demonstrating an inverse relationship between the space variable and $u(t)$. **Figures 4 and 5** represent the curve solutions for Problems 4 and 5, both spanning the time interval $t = [-1,1]$. For both problems, the solutions indicate that $u(t)$ increases with increasing space variable. This trend is similar to that observed in Problem 1, showing a positive correlation between the space variable and $u(t)$. **Figure 6** displays the curve solution for Problem 6, also within the time interval $t = [-1,1]$. The graph shows that $u(t)$ increases as the space variable increases, consistent with the trends observed in Figures 4 and 5. Each figure provides insights into how the function $u(t)$ behaves over different intervals and under different conditions, and the observed trends can be useful for understanding the underlying properties of the problems being analyzed.

Conclusions

Finally, we introduced a new transformation called the HK transformation. Properties such as linearity, scaling, convolution and shifting have been proven. The HK transformation has been used to analyze Volterra integral equations for the first kind and ordinary differential equations of both first and higher orders. The results are demonstrated through various problems and examples. In the future, this transformation will be applied to analyze additional models and fractional differential equations.

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