ZANCO Journal of Pure and Applied Sciences
The official scientific journal of Salahaddin University-Erbil
https://zancojournals.su.edu.krd/index.php/JPAS

ISSN (print):2218-0230**, ISSN (online):** 2412-3986**, DOI: http://dx.doi.org/10.21271/zjpas**

RESEARCH PAPER

Some results on S-numerical range of operator matrices

Berivan Faris Azeez [1], Ahmed Muhammad [2]

[1] Department of Mathematics, Faculty of Science and Health, Koya University, Koya، KOY45, Kurdistan Region- F.R. Iraq.
[2] Department of Mathematics, College of Science, Salahaddin University-Erbil, Kurdistan Region, Iraq

A B S T R A C T:
A linear operator on a Hilbert space may be approximated with finite matrices by choosing an orthonormal basis of the Hilbert space. In this paper, we found an approximation of the S-numerical range of bounded and unbounded operator matrices by variation methods. Applications to Hain-L$\ddot{u}$st operator and Stokes operator are given.

KEY WORDS: S-numerical range; projection method; Schr$\ddot{o}$dinger operator; Hain-L$\ddot{u}$st operator; Stokes operator.
DOI: http://dx.doi.org/10.21271/ZJPAS.32.3.7
ZJPAS (2020) , 32(3);57-63 .

1.INTRODUCTION :

Suppose $\mathcal{H}$ is a Hilbert space, with a scalar product $\langle\ ,\ \rangle$ and let S be a bounded self-adjoint operators. For an (possibly) unbounded linear operators $A: D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ we define

$$W_S^{\pm}(A) = \left\{\frac{\langle SAx, x\rangle}{\langle Sx, x\rangle} : x \in D(A), \langle Sx, x\rangle = \mp 1\right\}, \qquad (1)$$

where $D(.)$ denotes the domain. The sets $W_S^{\pm}(A)$ generalize the well-known and widely used notation of classical numerical range

$$W(A) = \{\langle Ax, x\rangle : x \in D(A), ||x|| = 1\}. \qquad (2)$$

By the well-known Toeplitz-Hausdorff Theorem (Hausdorff, 1919, Toeplitz, 1918). The set *W(A)* is convex. This set has been examined extensively see(Gustafson and Rao, 1997, R.A.Horn and C.R.Johnson, 1991) and has a lot of applications in functional analysis, operator theory, numerical analysis, perturbation theory, quantum mechanics see(Bebiano and Providência, 1998, Gustafson and Rao, 1997, Halmos, 2012), and the references therein.

There are many results concerning the interplay between the algebraic and analytic properties of an operator and the geometrical properties of its numerical range. Likewise, the indefinite numerical range motivated the interest of researchers see(Bebiano et al., 2008, Gustafson and Rao, 1997, Halmos, 2012, Li et al., 1996, Muhammad, 2005b, Muhammad, 2005a, N.Bebiano et al., 2004)): which in particular have investigated these relations in the Krein space setting. Although sharing some analogous properties with the classical numerical range, has a quite different behavior. Unlike the numerical range $W_S(A)$
is not convex. On the other hand it is neither closed nor bounded (Li et al., 1996).

We also define the related sets

$$W_S^{+}(A) = \left\{\frac{\langle SAx, x\rangle}{\langle Sx, x\rangle} : x \in D(A), \langle Sx, x\rangle = 1\right\}, \qquad (3)$$

And

$$W_S^{-}(A) = \left\{\frac{\langle SAx, x\rangle}{\langle Sx, x\rangle} : x \in D(A), \langle Sx, x\rangle = -1\right\}, \qquad (4)$$

* **Corresponding Author:**
Berivan Faris Azeez
E-mail: berivan.faris@koyauniversity.org
Article History:
Received: 02/12/2019
Accepted: 20/01/2020
Published: 15/06 /2020

It is well-known that each of the sets $W_S^+(A)$ and $W_S^-(A)$ is convex set and, as $W_S(A) = W_S^+(A) \cup W_S^-(A)$, $W_S(A)$ decomposes into at most two convex subsets. In (Bebiano et al., 2004) boundary generating curves, corners and computer generation of the Krein space numerical range are investigated, in (Bebiano et al., 2005, Li et al., 1996, Nakazato et al., 2011)relations between the sets $W_S^+(A)$ and $W_S^-(A)$ are discussed.

The set $W_S(A)$ and $W_S^{\pm}(A)$ have been investigated. When S is a nonsingular indefinite Hermitian matrix, some authors use $W_S(A)$ or $W_S^+(A)$ as the definition for a numerical range of A associated with the indefinite inner product $\langle u,u\rangle_S = \langle Su,u\rangle$. We list some basic properties of the S-numerical range that follows easily from the definition.

In this note we see how to compute $W_S(A)$ by projection methods, which reduce the problem to that of computing the S-numerical range of a (finite) matrix and block matrix.

The paper is organized as follows. In Section 2.1 and 2.2 some theoretical results are investigated dealing with the approximation of S-numerical range for a (possibly) unbounded operators using projection method. In Section 3, applying these results to compute the S-numerical range of operators.

1.1 Definition and Results

We initiate this subsection with a basic concept in functional analysis, the core of an operator, which will be utilized further in the remainder. For this reason and also for the sake of completeness, we remind the reader of the following well-known definitions.

Definition 1.1. [(Kato, 2013), p.166] Let A be an operator on a Hilbert space H. The set $C \subseteq D(A)$ is a core of A if for any $x \in D(A)$ there exist $x_k \in C$ such that $||x_k - x|| \longrightarrow 0$ and $||Ax_k - Ax|| \longrightarrow 0$.

The following easy observation will be useful, and its proof is similar to the proof of [(Tahiri, 2015),Theorem 2.4.12].

Theorem 1.2. If A is positive (negative) definite and S is indefinite, then $W_S(A)$ is the union of two disjoint unbounded intervals.

Lemma 1.3. [(Kato, 2013), Problem 5.16] If A is a bounded and closed operator, then any linear submanifold C of $D(A)$ dense in $D(A)$ is a core of A.

Proof Suppose that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k, \dots)^t \in \mathcal{D}(A)$, then there exist a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k, 0,0,0)^t \in C$ such that $||\alpha - \alpha_k|| = \sum_{j=k+1}^{\infty} |\alpha_j|^2 \longrightarrow 0$, and $||A\alpha - A\alpha_k|| \le ||A||||\alpha - \alpha_k|| \longrightarrow 0$.Thus C is a core of A.

2 S-numerical range approximation using projection methods

We use the following conventions. For any closed subspace $V \subset H$ we denote by P_V the orthogonal projection in H onto V. For a linear operator A, if $V \subset D(A)$ then $A_V := P_V A|_V$ denotes the compression of A to V.

Projection methods for accomplish a subset of the S-numerical range, under hypotheses. Only when one wishes to be sure of generating the whole of $W_S(A)$ it is important to make some extra assumptions. This section is devoted to the major results of the paper and it is divided into two subsections, since we distinguish between the estimation of the S-numerical range of a bounded and an unbounded operator.

2.1 Bounded linear operator

In the beginning of this section, we consider a bounded (linear) operator A on a complex Hilbert space H. We start with the following definitions.

Definition 2.1. Let A be an operator and $\{\emptyset_k : k \in \mathbb{N}\}$ be an orthonormal sequence of vectors H. For a fixed integer $k \ge 2$, the kxk matrices that arise from the operator A and the orthonormal vectors and $\{\emptyset_k : k \in \mathbb{N}\}$ are

$$\mathbb{A}_k = \begin{pmatrix} \langle A\phi_1, \phi_1\rangle \langle A\phi_1, \phi_2\rangle \cdots \langle A\phi_1, \phi_k\rangle \\ \langle A\phi_2, \phi_1\rangle \langle A\phi_2, \phi_2\rangle \cdots \langle A\phi_2, \phi_k\rangle \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \langle A\phi_k, \phi_1\rangle \langle A\phi_k, \phi_2\rangle \cdots \langle A\phi_k, \phi_k\rangle \end{pmatrix}, \quad (5)$$

that is the (p, r)-element of $\mathbb{A}_k$matrix is equal to $\langle A\phi_p, \phi_r\rangle$, for $p = 1,2, \dots, k$.

Definition 2.2. let S be a self-adjoint operator, and $\{\emptyset_k : k \in \mathbb{N}\}$ be an orthonormal sequence of vectors H. For a fixed integer $k \ge 2$, the kxk matrices that arise from the operators S and the orthonormal vectors $\{\emptyset_k : k \in \mathbb{N}\}$ are

$$\mathbb{S}_k = \begin{pmatrix} \langle S\phi_1, \phi_1\rangle \langle S\phi_1, \phi_2\rangle \cdots \langle S\phi_1, \phi_k\rangle \\ \langle S\phi_2, \phi_1\rangle \langle S\phi_2, \phi_2\rangle \cdots \langle S\phi_2, \phi_k\rangle \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \langle S\phi_k, \phi_1\rangle \langle S\phi_k, \phi_2\rangle \cdots \langle S\phi_k, \phi_k\rangle \end{pmatrix}, \quad (6)$$

that is the (p, r)-element of $\mathbb{S}_k$ matrix is equal to $\langle S\phi_p, \phi_r\rangle$, for $p = 1,2, \dots, k$.

Theorem 2.3. Let A be a bounded operator in a Hilbert space H and S be self-adjoint operator. Let $\{l_k : k \in \mathbb{N}\}$ be a nested family of subspaces in H, given by $k = span\{\phi_1, \phi_2, \dots, \phi_k\}$, where $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal basis of H. Consider kxk matrices $\mathbb{A}_k$ and $\mathbb{S}_k$in Eq. (5) and Eq. (6) respectively. Then $W_{\mathbb{S}_k}(\mathbb{A}_k) \subseteq W_s(A)$.

Proof Define an isometry $i: l_k \longrightarrow \mathbb{C}^k$ by $i(\alpha_1\phi_1, \alpha_2\phi_2, \dots, \alpha_k\phi_k) \coloneqq (\alpha_1, \alpha_2, \dots, \alpha_k)$. Suppose that $\lambda \in W_{\mathbb{S}_k}(\mathbb{A}_k)$, then there exist an S-unit vector $\alpha \in \mathbb{C}^k$, with $||x|| = 1$ such that $\lambda = \langle S\mathbb{A}_k\alpha, \alpha\rangle$. Choose $\xi \in l_k$, such that $i(\xi) = \alpha$ and $||\xi|| = 1$. Then a direct computation shows that $\lambda = \langle A\xi, \xi\rangle_s$ where $= \sum_{j=1}^{k} \alpha_j\phi_j$. Thus $\lambda \in W_s(A)$.

Proposition 2.4. Let $\{l_k : k \in \mathbb{N}\}$ and $\mathbb{A}_k$be as in Theorem 2.3 and let S be a self-adjoint operator. Given $r = 1,2, \dots$ then $W_{\mathbb{S}_k}(\mathbb{A}_k) \subseteq W_{\mathbb{S}_k}(\mathbb{A}_{k+r})$.

Proof This is an instant consequence of the fact that $\mathbb{C}^k$ is a subspace of $\mathbb{C}^{k+1}$. In detail

If λ is in $W_{\mathbb{S}_k}(\mathbb{A}_k)$ then there exist an S-unit vector $\alpha \in \mathbb{C}^k, ||\alpha|| = 1$ such that $\lambda = \langle S\mathbb{A}_k\alpha, \alpha\rangle$ where α can be extended to vectors in $\mathbb{C}^{k+1}$ say $\hat{\alpha}$ whose (k+1)th-components are zero. It is easy to see that $\langle S\mathbb{A}_k\alpha, \alpha\rangle = \langle S\mathbb{A}_{k+1}\alpha, \alpha\rangle$ and the result follows.

The following theorem stands as both a generalization and application of Theorem 2.3 and estimates the S-numerical range of a bounded operator by infinite union of S-numerical ranges of its suitable projection.

Theorem 2.5. Let A be a bounded operator on a Hilbert space H. Let S be a self-adjoint operator.
Also $\{l_k : k \in \mathbb{N}\}$, $\mathbb{A}_k$ and $\mathbb{S}_k$ be as in Theorem 2.3, then $W_S(A) = \overline{\bigcup_{l=2}^{\infty} W_{\mathbb{S}_k}(\mathbb{A}_k)}$.

Before proving this theorem, we require the following lemma.

Lemma 2.6. Let A be a bounded operator on the Hilbert space H. Let $(l_k)_{k\in\mathbb{N}}$ be a family of spaces in H with $l_k = span\{\phi_1, \phi_2, \dots, \phi_k\}$, where $\langle\phi_i, \phi_j\rangle = \delta_{jk}$. Then $\langle A_k\underline{\tilde{\alpha}}_k, \underline{\tilde{\alpha}}_k\rangle_{S,\mathbb{C}^k} = \langle A_k\underline{\tilde{\alpha}}_k, \underline{\tilde{\alpha}}_k\rangle_{S,H}$, where A_k is the sub matrix of the finite matrix of inner products $\langle\phi_i, \phi_j\rangle, 1 \le i, j < \infty, \underline{\tilde{\alpha}}_k = \alpha_1\emptyset_1, \alpha_2\emptyset_2, \dots, \alpha_k\emptyset_k$ and $\alpha_k = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{C}^k$.

Proof Since we know $(A_k)_{pq} \coloneqq \langle A\phi_p, \phi_q\rangle_S$, for each p, q. Then a simple computation shows that $\langle A_k\underline{\tilde{\alpha}}_k, \underline{\tilde{\alpha}}_k\rangle_{S,\mathbb{C}^k} = \sum_{q=1}^{k}\sum_{q=1}^{k}(SA_k)_{pq}\, \alpha_p\bar{\alpha}_q = \langle A\underline{\tilde{\alpha}}_k, \underline{\tilde{\alpha}}_k\rangle_{S,H}$

Proof [Proof of Theorem 2.5.] Using the preceding theorem it suffices to show that $W_S(A) \subseteq \overline{\bigcup_{l=2}^{\infty} W_{\mathbb{S}_k}(\mathbb{A}_k)}$. Let $\lambda \in W_S(A)$, then there exist an S-unit vector α∈H, with $||x|| = 1$ and $< \alpha, \alpha >_S = 1$ such that $\lambda = \frac{<A\alpha,\alpha>_S}{<\alpha,\alpha>_S}$. Suppose that $\mathcal{C} = \text{Span}\{\emptyset_1, \dots, \emptyset_k, \dots\}$ is a linear span of a countable of infinity orthonormal elements of H by Problem 1.1, $\mathcal{C}$ is a core of A. Thus by Definition 1.1 there exists a sequence $\alpha_1, \alpha_2, \dots$ with each $\alpha_k \in \mathbb{C}^k$given by $\alpha_k = P_k\alpha$, where $P_k: H \to l_k$ is orthogonal projection such that $||\alpha - \alpha_k|| \to 0$ and $||A\alpha - A\alpha_k|| \le ||A||||\alpha - \alpha_k|| \to 0$, Then for each $k \in \mathbb{N}$ choose $\lambda_k = \frac{<A_k\tilde{\alpha}_k,\tilde{\alpha}_k>_{\mathbb{S}_k}}{<\tilde{\alpha}_k,\tilde{\alpha}_k>_{\mathbb{S}_k}}$. Lemma 2.6 shows that there exists $\lambda_k \in W_{\mathbb{S}_k}(\mathbb{A}_k)$ such that $|\lambda_k - \lambda| \to 0$ as $k \to \infty$; hence $\overline{W_S(A)} \subseteq \overline{\bigcup_{k=1}^{\infty} W_{\mathbb{S}_k}(\mathbb{A}_k)}$.

2.2 Unbounded linear operators

We investigate the S-numerical range of an unbounded linear operator $A: \mathcal{D}(A) \subset H \to H$ where $\mathcal{D}(A)$ is the domain of A extending the result of the first subsection.

Theorem 2.7. Let $A: \mathcal{D}(A) \subset H \to H$ be an unbounded operator a on a Hilbert space H.
Let S be a self-adjoint operator. Let $\{l_k : k \in \mathbb{N}\}$ be a nested family space of $\mathcal{D}(A)$ given by $l_k = span\{\phi_1, \phi_2, \dots, \phi_k\}$, where $\{\emptyset_k : k \in \mathbb{N}\}$ is an orthonormal basis of H. Consider k x k matrices $\mathbb{A}_k$ and $\mathbb{S}_k$ in Eq. (5) and Eq. (6) respectively in Theorem 2.3. then $W_{\mathbb{S}_k}(\mathbb{A}_k) \subseteq W_S(A)$.

Proof Define an isometry $i: l_k \to \mathbb{C}^k$ by $i(\alpha_1\emptyset_1 + \alpha_2\emptyset_2 + \dots + \alpha_k\emptyset_k) \coloneqq (\alpha_1, \alpha_2, \dots, \alpha_k)$. Suppose that $\lambda \in W_S(\mathbb{A}_k)$. Then there exist an S-unit vector $\alpha\in\mathbb{C}^k$, with $< \alpha, \alpha >_S$=1, and λ is an eigenvalue of $(\mathbb{A}_j)_{\alpha,\alpha} \coloneqq < \mathbb{A}_k\alpha, \alpha >_S$. Choose $x \in l_k$, such that i(x) = α and $||x|| = 1$. In a good view of Lemma 2.6 this immediately gives $(\mathbb{A}_k)_{\alpha,\alpha} \coloneqq < \mathbb{A}_k\alpha, \alpha >_S$, so $\lambda \in W_S(A)$.

Proposition 2.8. In notation of Theorem 2.3, $(W_S(\mathbb{A}_k))_{k=1}^{\infty}$is a nested sequence, and let S be a self-adjoint operator. Then $W_{\mathbb{S}_k}(\mathbb{A}_k) \subseteq W_{\mathbb{S}_{k+1}}(\mathbb{A}_{k+1})$ for $k = 1,2, \dots$

Proof This is an instant sequence of the fact that $\mathbb{C}^k$ is the subspace of $\mathbb{C}^{k+1}$. In detail: if $\lambda\, is\, in\, W_{\mathbb{S}_k}(\mathbb{A}_k)$ then there exist an S-unit vector

$\alpha \in \mathbb{C}^k$, with $||\alpha|| = 1$ and $< \alpha, \alpha >_S = 1$ such that $\lambda = < \mathbb{A}_k \alpha, \alpha >_{\mathbb{S}_k}$ and α can be extended to vectors in $\mathbb{C}^{k+1}$, say $\hat{\alpha}$ whose $(k+1)-^{th}$ components are zero. It is easy to see that $< \mathbb{A}_k \alpha, \alpha >_{\mathbb{S}_k} = < \mathbb{A}_{k+1} \alpha, \alpha >_{\mathbb{S}_{k+1}}$ and the result follows.

Theorem 2.9. Let A, S, $\mathbb{A}_k$, and $\mathbb{S}_k$ be as in Theorem 2.7. Let $\mathcal{C} = \text{Span}\{\emptyset_1, \dots, \emptyset_k, \dots\}$ be a core of A. Then $W_S(A) = \overline{\cup_{k=1}^{\infty} W_{\mathbb{S}_k}(\mathbb{A}_k)}$.

Proof In the view of Theorem 2.7 it therefore now suffices to show that $W_S(A) = \overline{\cup_{k=1}^{\infty} W_{\mathbb{S}_k}(\mathbb{A}_k)}$. Let $\lambda \in W_S(A)$, then there exist an S-unit vector $x \in \mathcal{D}(A)$, with $||x|| = 1$ and $< x, x >_S = 1$ such that $\lambda = \frac{<Ax,x>_S}{<x,x>_S}$, since $\mathcal{C}$ is a core of A. There exist a sequence $(x_k)_{k=1}^{\infty}$ with each $x_k \in \text{Span}\{\emptyset_1, \dots, \emptyset_{S,k}\}$ for some $S_k > 0$, such that $||x - x_k|| \to 0$ and $||Ax - Ax_k|| \to 0$. Fix $k > 0$. Let $\text{i}: \text{Span}\{\emptyset_1, \dots, \emptyset_{S,k}\} \to \mathbb{C}^{S,k}$, be the standard isometrics as in the proof of theorem (2.3). Define $\hat{\alpha}_k \in \mathbb{C}^{S,k}$ by $\hat{\alpha}_k = i(x_k)$). Consider the $S_k \times S_k$ matrix $\mathbb{A}_{S,k}$ that is the *(p,r)*-element of $\mathbb{A}_{S,k}$ matrix is equal to $< A\emptyset_p, \emptyset_r >_S$ for $p, r = 1,2, \dots, k$. Then for each $k \in \mathbb{N}$ choose $\lambda_k = < A\hat{\alpha}_k, \hat{\alpha}_k >_{\mathbb{S}_k}$, Lemma 2.6 shows that there exists $\lambda \in W_{\mathbb{S}_k}(\mathbb{A}_k)$ such that $||\lambda - \lambda_k|| \to 0$ as $k \to \infty$. In view of Proposition 2.8 this immediately gives that $\lambda \in \overline{\cup_{k=1}^{\infty} W_{\mathbb{S}_k}(\mathbb{A}_k)}$.

3 Numerical experiments on a matrix differential operator

In this section, we will give and illustrate some examples based on a Schrödinger operator, multiplication operator, Hain-Lüst operator and Stocks operator to illustrate the theorems proved. The computations were performed in MATLAB.

3.1 S-numerical range of Schrödinger operator

3.2 Example 1

In the Hilbert space $H := L_2(0,1)$, we introduce the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q, \qquad (7)$$

(with bounded potential q) and the domain of L is given by

$$D(L) = \{u \in H^2(0,1): u(0) = 0 = u(1)\},$$

and let $S: L_2(0,1) \to L_2(0,1)$ be a multiplication operator defined on $L_2(0,1)$ by

$$(Sf)(t) = u(t)f(t), \qquad (8)$$

where $u(t) = 20t - 25$, $u \in C[0,1]$ and $f \in L_2(0,1)$, and the domain of S is given by $D(S) = L_2(0,1)$.

Remark 1.

(i) For this example, because L is self-adjoint and bounded below with purely discrete spectrum, the eigenvalues of L are given by

$$\lambda_n := \inf_{L \subset D(L)} \sup_{y_1 \in L,\ dim\, L = n,\ y \neq 0} g(y)$$

where g is the Rayleigh functional (Murnaghan, 1932),

$$g(y) := \frac{\langle Ly, y \rangle}{\langle y, y \rangle}, \qquad y \in D(L), y \neq 0.$$

Hence

$$\pi^2 = \lambda_1 := \inf_{y \in D(L),\ y \neq 0} g(y) \qquad (9)$$

(ii) We assume that the real valued potential q = 0, because if the operator L included a potential, for instance, then its eigenfunctions would not generally be explicitly computable. So still $-\frac{d^2}{dx^2}$ is equipped with Dirichlet boundary conditions on [0, 1]. It is obvious the eigenvalues and normalized eigenfunctions for the operator L in $L^2[0,1]$ are

$$\lambda_j = j^2\pi^2, \phi_j(x) = \sqrt{2}\sin(j\pi x), \quad j = 1,2,3, \dots \qquad (10)$$

under the setting $p(x) = 0$.

(iii) In Eq.(7), and Eq.(8) it is not difficult to see that, the linear span $C = \{\phi_1, \phi_2, \dots\}$ is a core of each L, and S respectively. Where $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal basis in $L^2(0,1)$.

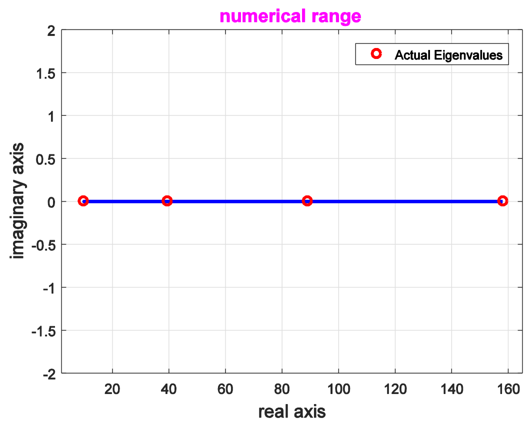
(iv) We may use the eigenfunctions in Eq. (10) as basis elements for discretization of the type discussed in section 2.1, forming the matrix elements $\langle L\phi_k, \phi_j \rangle$, and $\langle S\phi_k, \phi_j \rangle$ and consider the infinite operator matrices $\mathcal{Q} = \langle L\phi_k, \phi_j \rangle$ and $\hat{\mathcal{Q}} = \langle S\phi_k, \phi_j \rangle$. The matrices $\mathbb{A}_k$ and $\mathbb{S}_k$ defined in Eq .(5) and Eq. (6) are obtained by leading sub-matrices of the $\mathcal{Q}$, and $\hat{\mathcal{Q}}$ with the appropriate dimensions. Observe that

$$\langle L\phi_k, \phi_j \rangle_S = diag\{\pi^2, 4\pi^2, 9\pi^2, \dots\}, \qquad (11)$$

and
$$\langle S\phi_k, \phi_j\rangle = 20\int_0^1 2t\sin(j\pi x)\sin(j\pi x) - 25\delta_{jk}dx \quad (12)$$

(v) If we assume that the Hermitian matrix $\mathbb{S}_k \in M_n$ is non-singular, then it is not a restriction to consider the matrix $J = I_k \oplus -I_k$ instead of $\mathbb{S}_k$, in the definition of the S-numerical range, where I_k is the identity matrix.

Figure (1) shows attempts to compute $W_{\mathbb{S}_k}(\mathbb{A}_k)$ for various k and also some attempts to estimate these sets by qualitative means, using existing theorems from the literature as well as the theorems proved above.



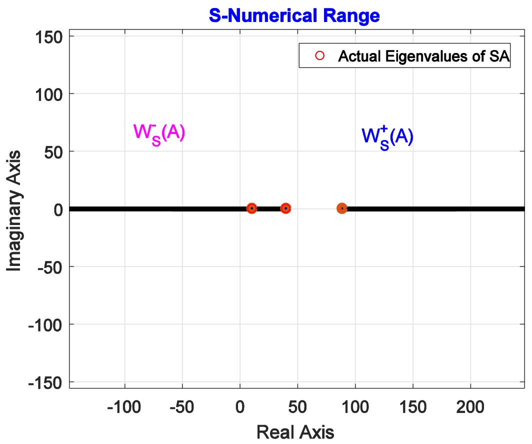


Figure 1: On the left-hand side, estimation of numerical range of $\mathbb{A}_k$ for k = 4. While for the right-hand side, estimation of $W_{\mathbb{S}_k}(\mathbb{A}_k)$ for $k = 4$.

Remark 2.

(i) It is clear the numerical range of $\mathbb{A}_k$ for $k = 4$ is equal to $[\pi^2, 16\pi^2]$.

(ii) In this example $\mathbb{A}_k$ is positive definite and $\mathbb{S}_k$ is indefinite, then according to Theorem 1.2 $W_{\mathbb{S}_k}(\mathbb{A}_k)$ is the union of two disjoint unbounded intervals $(-\infty, 9\pi^2] \cup [16\pi^2, \infty)$.

3.4 S-numerical range of Hain-L$\ddot{u}$st operator

Example 2

In the Hilbert space $L_2^2(0,1) := L^2(0,1) \oplus L^2(0,1)$ we introduce the matrix differential operator
$$A := \begin{pmatrix} \tilde{L} & w(x) \\ \widetilde{w}(x) & u(x) \end{pmatrix}, \quad (13)$$
on the domain
$$\mathcal{D}(A) := \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1 \in (H_0^1(0,1) \cup H^2(0,1)) \text{ and } y_2 \in L_2(0,1) \right\}$$
Where $\tilde{L}$ is the Sturm-Liouville operator $\tilde{L}y = -y''$ with a Dirichlet boundary conditions, $w = \widetilde{w} = 1$ and $u = 20x - 25$. This operator was introduced by Hain and L¨ust in application to problems of magneto hydrodynamics (Hain and Lust, 1958), and the problems of this type were studied in (Langer et al., 1990), (Adamjan and Langer, 1995) and (Langer and Tretter, 1998).

Now from the matrix elements $< \tilde{L}\phi_k, \phi_j >$, $< \widetilde{w}\phi_k, \phi_j >$, $< w\phi_k, \phi_j >$, $< u\phi_k, \phi_j >$. With respect orthonormal basis in Eq. (10) and consider the infinite block operator matrix.
$$Q = \begin{pmatrix} < \tilde{L}\phi_k, \phi_j >, & < \widetilde{w}\phi_k, \phi_j > \\ < w\phi_k, \phi_j > & < u\phi_k, \phi_j > \end{pmatrix}.$$
The matrix $\mathbb{A}$ defined in (5) is obtained by taking leading sub-matrix of the block of Q, with appropriate dimensions. Observe that

$< \tilde{L}\phi_k, \phi_j > = \text{diag}\{\pi^2, 4\pi^2, 9\pi^2, \dots\}$,

$< \widetilde{w}\phi_k, \phi_j > = diag\{1, 1, 1, \dots\}$,

$< w\phi_k, \phi_j > = diag\{1, 1, 1, \dots\}$,

$< u\phi_k, \phi_j > = 20\int_0^1 x\sin(k\pi x)\sin(j\pi x)\, dx - 25\delta_{k,j}$.

Let S be a self-adjoint operator
$$S = \begin{pmatrix} \tilde{L} & \widetilde{w} \\ w & u \end{pmatrix}.$$
Where $\tilde{L}$ is the Sturm-Liouville operator $\tilde{L}y = -y''$ with Dirichlet boundary conditions, $w = \widetilde{w} = 1$ and $z = e^x$.

The domain of S in this case is given by
$$\mathcal{D}(A) := \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1 \in (H_0^1(0,1) \cup H^2(0,1)) \text{ and } y_2 \in L_2(0,1) \right\}.$$
By the same argument the matrix elements $< \tilde{L}\phi_k, \phi_j >$, $< \widetilde{w}\phi_k, \phi_j >$, $< w\phi_k, \phi_j >$,

$< u\phi_k, \phi_j >$. With respect orthonormal basis in Eq. (10) and consider the infinite block operator matrix.

$$\tilde{Q} = \begin{pmatrix} < \tilde{L}\phi_k, \phi_j >, & < \tilde{w}\phi_k, \phi_j > \\ < w\phi_k, \phi_j > & < z\phi_k, \phi_j > \end{pmatrix}.$$

The matrix $\mathbb{S}_k$ defined in Eq. (6) are obtained by taking sub-matrix of the block of Q, with appropriate dimensions. Observe that

$\widetilde{< L\phi_k, \phi_j >} = \text{diag}\{\pi^2, 4\pi^2, 9\pi^2, \ldots\}$,

$< \tilde{w}\phi_k, \phi_j >= diag\{1,1,1,\ldots\}$,

$< w\phi_k, \phi_j \geq\ diag\{1,1,1,\ldots\}$,

$< u\phi_k, \phi_j >= \int_0^1 \sin(k\pi x)\sin(j\pi x)\,dx$.

Remark 3. It is not difficult to see that the subspace $\mathcal{C}_1 \coloneqq \mathcal{C}_{\tilde{L}} \oplus \mathcal{C}_u \subset \mathcal{D}(A) = \left(D(\tilde{L}) \cap D(\tilde{w})\right) \oplus \left(D(w) \cap D(u)\right)$, is a core of A also.

$\mathcal{C}_1 \coloneqq \mathcal{C}_{\tilde{L}} \oplus \mathcal{C}_z \subset \mathcal{D}(S) = \left(D(\tilde{L}) \cap D(\tilde{w})\right) \oplus \left(D(w) \cap D(z)\right)$, is a core of S.

Figure (2) shows attempts to compute $W_{\mathbb{S}_k}(\mathbb{A}_k)$ for various k and also some attempts to estimate these sets by qualitative means, using existing theorems from the literature as well as the theorems proved above.

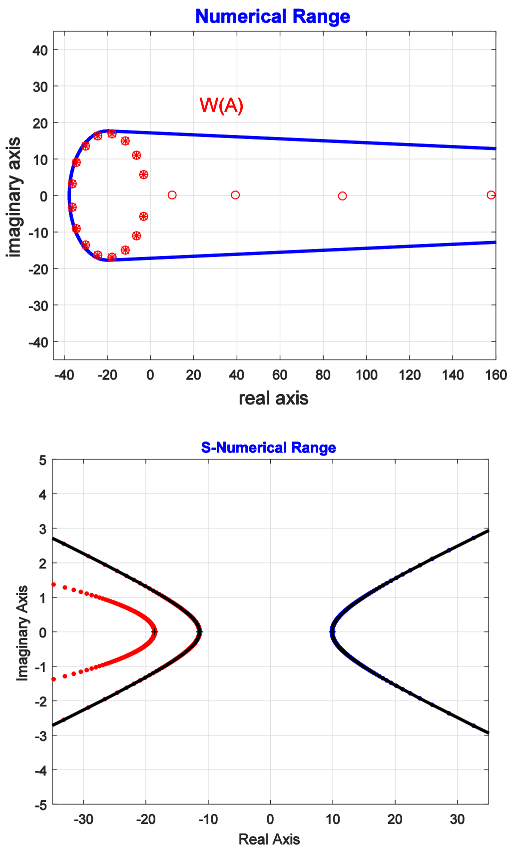


Figure 2: On the left-hand side, estimation of numerical range of $\mathbb{A}_k$ for k = 18. While for the right-hand side, estimation of $W_{\mathbb{S}_k}(\mathbb{A}_k)$ for $k = 4$.

Remark 4.

(i) In order to understand the right-hand side result in Figure 2 it is helpful to find an analytical. Estimate for *W(A)*. Let $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{D}(A)$, $||\vec{y}|| = 1$, and let $\lambda =< A\vec{y}, \vec{y} > = < -D^2 y_1, y_1 > +< y_2, y_1 > +< y_1, y_2 > +< z y_2, y_2 >= \int_0^1 |y_1'|^2 + 2\Re\left(\int_0^1 y_1\overline{y_2}\right) + \int_0^1 z|y_2|^2\,dx$ (14)

Equation (14) gives us an estimate for the first term on the right hand side of (14),

$$\int_0^1 |y_1'|^2 \geq \pi^2 \int_0^1 |y_1|^2 \quad (15)$$

For the second term on the right hand side of (14), the Cauchy Schwarz inequality and Youngs inequality yield

$$2\Re\left(\int_0^1 y_1\overline{y_2}\right) \geq -\left(\int_0^1 |y_1|^2 + |y_2|^2\right)dx. \quad (16)$$

Also third term of the right hand side of equation (14) satisfies

$$\Re\left(\int_0^1 z|y_2|^2\,dx\right) \geq \inf \Re(z)\left(\int_0^1 |y_2|^2\,dx\right). \quad (17)$$

Hence from Equations (15), (16), (17) we get that

$$\Re(\lambda) \geq \pi^2 \int_0^1 |y_1|^2\,dx - 1 + \inf \Re(z) \int_0^1 |y_2|^2\,dx. \quad (18)$$

This simplifies to

$$\Re(\lambda) \geq \pi^2 ||y_1||^2 - 1 + \left(1 - ||y_1||^2\right) inf\ \Re(z)$$

$$= \pi^2 - \inf \Re(z) ||y_1||^2 + \inf \Re(z) - 1.$$

This yields

$$\Re(\lambda) = \begin{cases} \inf \Re(z) - 1, & if\ \pi^2 - \inf \Re(z) \geq 0; \\ \pi^2 - 1, & if\ \pi^2 - \inf \Re(z) < 0. \end{cases}$$

For our example these yield $\Re(\lambda) \geq -39$.

To estimate $Im(\lambda)$ observe that

$$Im(\lambda) = \int_0^1 (Im(z))|y_2|^2$$

$$\leq \sup_{x\in[0,1]} (Im(z)) \int_0^1 |y_2|^2 dx \leq 18, \quad (19)$$

and

$$Im(\lambda) = \int_0^1 (Im(z))|y_2|^2$$

$$\leq \sup_{x\in[0,1]} (Im(z)) \int_0^1 |y_2|^2 dx \geqslant -18, \quad (20)$$

This completes the estimates on *W(A)*.

(ii) On the other hand for the right-hand side, since the S-numerical range is in general neither bounded nor closed, it is difficult to generate an accurate computer plot of this set. For $\mathbb{A}_k \in M_k$ and $k > 2$, the description of $\mathbb{A}_k$ is complicated, so in our example

$W_{\mathbb{S}_k}(\mathbb{A}_k)$ is bounded by the hyperbola centered at (0,1) and The foci of the hyperbolas are the eigenvalues of $\mathbb{A}_k$.

4 Conclusions

Our results describes the practical difficulties that related with the S-numerical ranges of operator matrices and block operator matrices of differential operators, even so good theoretical outcomes are available to underpin the approximation procedure. Completely analytic approaches are important to understand while the numerical results are deceptive, and apparently numerical results should be deal with skepticism.

References:

ADAMJAN, V. M. & LANGER, H. 1995. Spectral properties of a class of rational operator valued functions. *Journal of Operator Theory***,** 259-277.

BEBIANO, N., LEMOS, R., DA PROVIDENCIA, J. & SOARES, G. 2004. On generalized numerical ranges of operators on an indefinite inner product space. *Linear and Multilinear Algebra,* 52**,** 203-233.

BEBIANO, N., LEMOS, R., DA PROVIDÊNCIA, J. & SOARES, G. 2005. On the geometry of numerical ranges in spaces with an indefinite inner product. *Linear algebra and its applications,* 399**,** 17-34.

BEBIANO, N. & PROVIDÊNCIA, J. O. D. 1998. Numerical ranges in physics. *Linear and Multilinear Algebra,* 43**,** 327-337.

BEBIANO, N., PROVIDIA, J. D., NATA, A. & SOARES, G. 2008. *Krein Spaces Numerical Ranges and their Computer Generation*, Electron. J. Linear Algebra,.

GUSTAFSON, K. E. & RAO, D. K. 1997. Numerical range. *Numerical Range.* Springer.

HAIN, K. & LUST, R. 1958. Zur Stabilität zylindersymmetrischer Plasmakonfigurationen mit Volumenströmen. *Zeitschrift für Naturforschung A,* 13**,** 936-940.

HALMOS, P. R. 2012. *A Hilbert space problem book*, Springer Science & Business Media.

HAUSDORFF, F. 1919. Der wertvorrat einer bilinearform. *Mathematische Zeitschrift,* 3**,** 314-316.

KATO, T. 2013. *Perturbation theory for linear operators*, Springer Science & Business Media.

LANGER, H., MENNICKEN, R. & MÖLLER, M. 1990. A second order differential operator depending nonlinearly on the eigenvalue parameter. *Oper. Theory Adv. Appl,* 48**,** 319-332.

LANGER, H. & TRETTER, C. 1998. Spectral decomposition of some nonselfadjoint block operator matrices. *Journal of Operator Theory***,** 339-359.

LI, C.-K., TSING, N.-K. & UHLIG, F. 1996. Numerical ranges of an operator on an indefinite inner product space. *Electronic Journal of Linear Algebra*.

MURNAGHAN, F. D. 1932. On the field of values of a square matrix. *Proceedings of the National Academy of Sciences of the United States of America,* 18**,** 246.

NAKAZATO, H., BEBIANO, N. & DA PROVIDÊNCIA, J. 2011. THE NUMERICAL RANGE OF LINEAR OPERATORS ON THE 2-DIMENTIONAL KREIN SPACE. *ELECTRONIC JOURNAL OF LINEAR ALGEBRA,* 22**,** 430-442.

R.A.HORN & C.R.JOHNSON 1991. *"Topics in matrix analysis"*, Cambridge University Press, New York.

TAHIRI, F. E. 2015. *Numerical Ranges of Linear Pencils. PhD thesis,* University of Coimbra.

TOEPLITZ, O. 1918. *Das algebraische analogon zu einem salze you fient*, Math. Z. Vol.2, (1918),187-197.

BEBIANO, N., PROVIDIA, J. D., NATA, A. & SOARES, G. 2008. *Krein Spaces Numerical Ranges and theirComputer Generation*, Electron. J. Linear Algebra,.

GUSTAFSON, K. E. & RAO, D. K. 1997. Numerical range. *Numerical Range.* Springer.

HALMOS, P. R. 2012. *A Hilbert space problem book*, Springer Science & Business Media.

LI, C.-K., TSING, N.-K. & UHLIG, F. 1996. Numerical ranges of an operator on an indefinite inner product space. *Electronic Journal of Linear Algebra*.

MUHAMMAD, A. M. S. 2005a. *Elliptical range of n-tuple operators on a complex Hilbert space*, Zanko Journal for Pure and Applied Science.

MUHAMMAD, A. M. S. 2005b. *Line segments of the boundary of numerical range,* Zanko Journal for Pure and Applied Science.

N.BEBIANO, LEMOS, R., PROVIDENCIA, J. D. & SOARES, G. 2004. On generalized numerical ranges of operators on an indenite inner product space, Linear and Multilinear Algebra, 52:203233. *Mathematische Annalen*.