ZANCO Journal of Pure and Applied Sciences
The official scientific journal of Salahaddin University-Erbil
https://zancojournals.su.edu.krd/index.php/JPAS

ISSN (print):2218-0230, **ISSN (online):** 2412-3986, **DOI:** **http://dx.doi.org/10.21271/zjpas**

RESEARCH PAPER

Darboux and Analytic First Integrals of Kingni–Jafari System with Only One Stable Equilibrium Point.

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A B S T R A C T:
In this paper, we illustrate by an evidence that the Kingni–Jafari differential system $\dot{u} = -w,\ \dot{v} = -u - w,\ \dot{w} = 3u - av + u^2 - w^2 - vw + b,$ where a and b are real parameters has no Darboux and rational first integrals for any value of a, b. Furthermore, we show that this system has no global C^1 first integrals for $a \in (0,3), b > 0$ and $3b - ab > a^2$. Also, an analytic first integral for some generic condition is studied of this system at the neighborhood of the equilibrium point $\left(0, \frac{b}{a}, 0\right)$.

KEY WORDS: Invariant Algebraic Surfaces, Darboux First Integral, Exponential Factor, Analytic First Integral.
DOI: http://dx.doi.org/10.21271/ZJPAS.32.3.1
ZJPAS (2020) , 32(3);1-9 .

1.INTRODUCTION :

Kingni and Jafari in (Kingni et al., 2014) proposed the simplest electronic circuit design. This electronic circuit consists of a resistors, AD633 multiplier, capacitors and operational amplifiers. This circuit can be considered by the following three-dimensional chaotic differential system

$$\begin{aligned}\dot{u} &= -w,\\ \dot{v} &= -u - w,\\ \dot{w} &= 3u - av + u^2 - w^2 - vw + b.\end{aligned} \tag{1}$$

has a rare equilibrium point $(0, \frac{b}{a}, 0)$ for $a \neq 0$. The study of chaotic System (1) is significant in physics and engineering applications, especially in circuit, control and communications.In (Wei et al., 2016), the authors proved that this system is the chaotic system with invisible attractors and that the stable equilibrium point can coexist with a strange attractor for specific parameters. Dynamics of the Kingni and Jafari system have explained via numerical simulations such as phase portraits, bifurcation diagrams and new cost function for parameter estimation. Wei et al (2016) have learned complex dynamical behaviors and topological structure of the system such as the dynamics of this system at infinity, periodic solutions, Hopf bifurcation and zero Hopf bifurcation. System (1) has been studied in the papers (Kingni et al., 2014 and Wei et al., 2016) but none of those papers mentions the integrability or non-integrability. In this paper, we investigate

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Article History:
Received: 08/08/2019
Accepted: 31/10/2019
Published: 15/06 /2020

first integrals of a Darboux and an analytic type of system (1).

Solutions of a differential system can be compared with the existing behavior of a system to make if the theoretical detailing of the system is accurate. This is interesting topic in the sciences. A Darboux integrability is a method to find a solution of a differential system, for more details see (Ollagnier, 1997, Christopher et al., 2007, Llibre and Valls, 2011a, Llibre and Valls, 2011b and Hussien and Amen, 2018).

2. PRELIMINARY RESULTS.

This section is started with a short overview of the integrability problem, the Darboux method and the auxiliary results which are given (Llibre and Zhang, 2002, Llibre and Zhang, 2010, Llibre and Valls, 2011b and Llibre and Zhang, 2012). To prove our important results, firstly we give some basic definitions and theorems as a background to this study.

The associated vector field to system (1) is define by

$$\chi = -w\frac{\partial}{\partial u} + (-u - w)\frac{\partial}{\partial v} + (3\,u - a\,v + u^2 - w^2 - v\,w + b)\frac{\partial}{\partial w}. \qquad (2)$$

Let D be an open subset of $\mathbb{C}^3$, a non-constant function $H: D \to \mathbb{C}$ is a first integral of the polynomial vector field χ on D if it is a constant on all orbits $\big(u(t), v(t), w(t)\big)$ of χ contained in D. Obviously, that H is called a first integral of χ on D if and only if

$$\chi(H) = -w\frac{\partial H}{\partial u} + (-u - w)\frac{\partial H}{\partial v} + (3\,u - a\,v + u^2 - w^2 - v\,w + b)\frac{\partial H}{\partial w} = 0. \qquad (3)$$

A local (global) first integral H is a first integral whose domain of definition is a neighborhood of an equilibrium point (whose domain of definition is $\mathbb{R}^3$) of system (1). We recall that H is an analytic (rational) first integral if it is an analytic (rational) function.

An equilibrium point (u_0, v_0, w_0) of system (1) is said to be an attractor if all eigenvalues λ_i of the Jacobian matrix of (1) at (u_0, v_0, w_0) have negative real parts.

Theorem 2.1. Routh-Hurwitz Criterion. The zero of $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ have negative real parts if and only if $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$.

We present the following results concerning with the non-existence of first integral, that we use later on, this is due to (Llibre et al., 2015).

Theorem 2.2. If system (1) has an equilibrium point (u_0, v_0, w_0) which is either repeller or attractor, then system (1) has no C^1 first integrals defined in a neighborhood at (u_0, v_0, w_0).

A Darboux theory of integrability has a best method to determine that systems have a first integral or not. Now, we will describe its some basic nations, for more deep information look at (Christopher and Llibre, 2000 and Llibre and Valls, 2011b)). Suppose that $f = f(u, v, w) \in \mathbb{C}[u, v, w]$, then $f = 0$ is said to be an invariant algebraic surface or it is called a Darboux polynomial of χ if there exist a polynomial $K_f \in \mathbb{C}[u, v, w]$ such that

$$\chi(f) = -w\frac{\partial f}{\partial u} + (-u - w)\frac{\partial f}{\partial v} + (3\,u - a\,v + u^2 - w^2 - v\,w + b)\frac{\partial f}{\partial w} = f\,K_f, \qquad (4)$$

we recall K_f is the cofactor of f and the degree of K_f here is at most 1.

Proposition 2.3. System (1) has a rational first integral if it has two different Darboux polynomials with the match cofactors.

We denote an exponential factor of system (1) by E which defined by a non-constant function of the form $E = e^{\frac{g}{f}}$ with greatest common divisor between g and f is equal to one. That means $(g, f) = 1$, where $g, f \in \mathbb{C}[u, v, w]$ and it is satisfied

$$\chi(E) = -w\frac{\partial E}{\partial u} + (-u - w)\frac{\partial E}{\partial v} + (3\,u - a\,v + u^2 - w^2 - v\,w + b)\frac{\partial E}{\partial w} = E\,L, \qquad (5)$$

for some polynomial $L = L(u, v, w) \in \mathbb{C}[u, v, w]$ of degree at most 1 which is called the cofactor of E.

Proposition 2.4. i) The function $E = e^{\frac{g}{f}}$ is an exponential factor of polynomial differential system (1) and f is a non-constant polynomial, then $f = 0$ is an invariant algebraic surface.
ii) Finally e^g can be an exponential factor, getting from the multiplicity of the infinity invariant plane.

Theorem 2.5. Darboux Theorem (Christopher and Llibre, 2000). Suppose that a polynomial vector field χ of degree d in $\mathbb{C}^3$ have p irreducible invariant algebraic surfaces $f_i = 0$ such that the f_i are pairwise relatively prime with cofactors K_i for $i = 1, \dots, p$ and q exponential factors $e^{\frac{g_j}{f_j}}$ together cofactors L_j for $j = 1, \dots, q$. There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0, \qquad (6)$$

if and only if the function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} \left(\left[e^{\frac{g_1}{f_1}} \right]^{\mu_1} \dots \left[e^{\frac{g_q}{f_q}} \right]^{\mu_q} \right), \qquad (7)$$

is the first integral of system (1).
The form (7) is called a Darboux first integral.
The following proposition is essential to prove the existence of an analytic first integral of system (1) which is due to (Llibre and Valls, 2008).
Proposition 2.6. (Llibre and Valls, 2008) . The 3-dimensioal linear differential system

$$\dot{U} = PU, \text{ where } U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \ \dot{U} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix},$$

has two independent first integrals which are given in the following cases

1. $F_1 = \frac{u^{\lambda_2}}{v^{\lambda_1}}$ and $F_2 = \frac{u^{\lambda_3}}{w^{\lambda_1}}$ if $P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, with $\lambda_i \in \mathbb{R}\backslash\{0\}$, $i = 1,2,3$.

2. $F_1 = \frac{w^{\lambda_1}}{u^{\lambda_2}}$ and $F_2 = w \exp\left(-\frac{\lambda_2 v}{u}\right)$ if $P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$, with $\lambda_i \in \mathbb{R}\backslash\{0\}$, $i = 1,2$.

3. $F_1 = \frac{u^2}{2u\,w - v^2}$ and $F_2 = u \exp\left(-\frac{\lambda v}{u}\right)$ if $P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 1 & \lambda_1 \end{pmatrix}$, with $\lambda_1 \in \mathbb{R}\backslash\{0\}$.

4. $F_1 = \frac{(u^2+v^2)^{\lambda}}{w^{2\alpha}}$ and $F_2 = \exp\left(-2\alpha \arctan(\frac{v}{u})\right) (u^2 + v^2)^{\beta}$ if $P = \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, with $\lambda, \ \alpha, \ \beta \in \mathbb{R}\backslash\{0\}$.

5. $F_1 = (u^2 + v^2)$ and $F_2 = \exp\left(-\lambda \arctan(\frac{v}{u})\right) w^{\beta}$ if $P = \begin{pmatrix} 0 & -\beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, with $\lambda, \ \beta \in \mathbb{R}\backslash\{0\}$.

3. MAIN RESULTS AND THEIR PROVING

In this part, the existence of rational first integrals (see Theorem 3.3), Darboux first integrals (see Theorem 3.5) and an analytic first integral (Theorem 3.8) are the main results of system (1) are described. Moreover, some other results relative to this topic are studied in this work such as a polynomial first integral, invariant algebraic surfaces, exponential factors and C^1 first integrals of system (1).
The following proposition is the first result in this work.
Proposition 3.1. System (1) has no polynomial first integrals.
Proof. Let $H = \sum_{i=1}^{n} H_i(u, v, w)$ be a polynomial first integral of system (1), where each H_i is a

homogeneous polynomial in its variables of degree i. By definition of first integral, we have

$$-w\,\frac{\partial}{\partial u}H+(-u-w)\frac{\partial}{\partial v}H+(3\,u-a\,v+u^2-\,w^2-v\,w+b)\frac{\partial}{\partial w}H=0. \qquad (8)$$

Computing the terms of degree $n+1$, we obtain

$$(\,u^2-\,w^2-v\,w)\frac{\partial}{\partial w}H_n(u,v,w)=0,$$

that is

$$H_n(u,v,w)=F_1(u,v),$$

where F_1 is a polynomial of variables u and v of degree n. Also, computing the terms of degree n in equation (8), we have

$$-w\,\frac{\partial}{\partial u}F_1(u,v)+(-u-w)\frac{\partial}{\partial v}F_1(u,v)+\left(3\,u-a\,v\right)\frac{\partial}{\partial w}F_1(u,v)+(u^2-\,w^2-v\,w\Big)\frac{\partial}{\partial w}H_{n-1}(u,v,w)=0,$$

this gives

$$H_{n-1}(u,v,w)=\frac{1}{\sqrt{4u^2+v^2}}\left(\left((2\,u-v)\left(\frac{\partial}{\partial v}F_1(u,v)\right)-v\,\left(\frac{\partial}{\partial u}F_1(u,v)\right)\right)\operatorname{arctanh}\left(\frac{v+2\,w}{\sqrt{4u^2+v^2}}\right)+\left(\left(-\frac{1}{2}\,\frac{\partial}{\partial u}F_1(u,v)-\frac{1}{2}\,\frac{\partial}{\partial v}F_1(u,v)\right)\ln(-u^2+w^2+v\,w)+F_2(u,v)\right)\sqrt{4u^2+v^2}\right).$$

Since $H_{n-1}(u,v,w)$ is a polynomial of degree $n-1$, then we have

$$\frac{\partial}{\partial u}F_1(u,v)+\,\frac{\partial}{\partial v}F_1(u,v)=0, \qquad (9)$$

and

$$(2u-v)\left(\frac{\partial}{\partial v}F_1(u,v)\right)-v\,\left(\frac{\partial}{\partial u}F_1(u,v)\right)=0, \qquad (10)$$

It is clear that the solution of equation (9) is

$$F_1(u,v)=F_3(v-u),$$

where F_3 is polynomial of variables u and v. Since, F_1 is the polynomial of degree n then it must be in the formula

$$F_1(u,v)=c\,(v-u)^n, \qquad (11)$$

where c is arbitrary constant. Putting (11) in (10), we obtain

$$c\,n\,(v-u)^n\,u=0,$$

This gives $c\,n=0$, then $c=0$ or $n=0$. If $c=0$, this implies that $F_1=0$, then $H_n(u,v,w)=0$, in this case system (1) has no polynomial first integrals. If $n=0$ then H is a constant function, this is trivial. This means that there is no a polynomial first integral of system (1).

Proposition 3.2. System (1) does not have invariant algebraic surfaces with non-zero cofactors.

Proof. Suppose that $f=\sum_{i=1}^{n}f_i(\,u,v,w)$ is an invariant algebraic surfaces of system (1) with the cofactor $K=k_0+k_1u+k_2v+k_3w$, where $k_i\in\mathbb{C}$ for $i=0,\dots,3$, and each f_i is a homogeneous polynomial in its variables of degree i. Assume that $f_n\neq 0$ for $n>1$, then by definition of invariant algebraic surface, we obtain

$$-w\,\frac{\partial}{\partial u}f+(-u-w)\frac{\partial}{\partial v}f+(3\,u-a\,v+\,u^2-w^2-v\,w+b)\frac{\partial}{\partial w}f=Kf. \qquad (12)$$

We first compute the terms of degree $n+1$ to obtain

$$(u^2-\,w^2-v\,w)\frac{\partial}{\partial w}f_n(u,v,w)=(k_1u+k_2v+k_3w)\,f_n(u,v,w). \qquad (13)$$

This gives

$$f_n(u,v,w)=G_1(u,v)\,(-u^2+\,w^2+v\,w)^{-\frac{k_3}{2}}e^{\frac{arc\tanh\left(\frac{2w+v}{\sqrt{4u^2+v^2}}\right)(2k_1u+2k_2v-k_3v)}{\sqrt{4u^2+v^2}}}, \qquad (14)$$

since $f_n(u,v,w)$ is a polynomial function, this implies that $k_1=0, k_2=-m$ and $k_3=-2m$ where $m\in\mathbb{N}\cup\{0\}$. Then equation (14) becomes

$$f_n(u,v,w)=G_1(u,v)\,(-u^2+\,w^2+v\,w)^m,$$

where G_1 is a polynomial of variables u and v of degree $n-2m$. Also, calculating the terms of degree n in equation (12), we take out

$$-w\,\left(\left(\frac{\partial}{\partial u}\,G_1(u,v)\right)(-u^2+\,w^2+v\,w)^m-\frac{2m\,u\,G_1(u,v)\,(-u^2+w^2+v\,w)^m}{-u^2+w^2+v\,w}\right)+(-u-w)\left(\left(\frac{\partial}{\partial v}G_1(u,v)\right)(-u^2+\,w^2+v\,w)^m+\right.$$

$$\frac{m\,w\,G_1(u,v)\,(-u^2+w^2+v\,w)^m}{-u^2+w^2+v\,w}\Big) + (3\,u - a\,v)\left(\frac{m\;(2\,w+v)\,G_1(u,v)\,(-u^2+w^2+v\,w)^m}{-u^2+w^2+v\,w}\right) + (u^2 - w^2 - v\,w)\,\left(\frac{\partial}{\partial w}\,f_{n-1}(u,v,w)\right) = k_0 G_1(u,v)\,(-u^2+\ w^2+v\,w)^m + (-mv-2mw) f_{n-1}(u,v,w),$$

this gives

$$f_{n-1}(u,v,w) = \left(-\frac{1}{2}\ln(-u^2+\ w^2+ v\,w)\,\left(\frac{\partial}{\partial u}\,G_1(u,v) + \frac{\partial}{\partial v} G_1(u,v)\right) + \frac{1}{(4u^2+v^2)^{\frac{3}{2}}}\left(4\,\left(2\,\left(u^2+\frac{1}{4}v^2\right)\left(u - \frac{1}{2}v\right)\left(\frac{\partial}{\partial u}\,G_1(u,v)\right) + \left(-u^2 v - \frac{1}{4}v^3\right)\left(\frac{\partial}{\partial u}\,G_1(u,v)\right) + G_1(u,v)\left((m + 2\,k_0)\,u^2 - \frac{1}{2}m\,u\,v + \frac{1}{2}k_0\,v^2\right)\right)\operatorname{arctanh}\left(\frac{v+2w}{\sqrt{4u^2+v^2}}\right)\right) + \left(\frac{4m\,G_1(u,v)\left(-\frac{7}{2}u^3+\left(\left(-\frac{1}{4}+a\right)v+\frac{1}{2}w\right)u^2-\frac{3}{4}\,\left(v-\frac{1}{3}w\right)u\,v+\frac{1}{4}(av+w)\,v^2\right)}{(4u^2+v^2)(u^2-w^2-v\,w)} + G_2(u,v)\right)(-u^2+\ w^2+v\,w)^m.$$

Since f_{n-1} is a polynomial then we have

$$\frac{\partial}{\partial u}\,G_1(u,v) + \frac{\partial}{\partial v} G_1(u,v) = 0, \qquad (15)$$

$$\frac{1}{(4u^2+v^2)^{\frac{3}{2}}}\left(4\,\left(2\,\left(u^2+\frac{1}{4}v^2\right)\left(u - \frac{1}{2}v\right)\left(\frac{\partial}{\partial u}\,G_1(u,v)\right) + \left(-u^2 v - \frac{1}{4}v^3\right)\left(\frac{\partial}{\partial u}\,G_1(u,v)\right) + G_1(u,v)\left((m + 2\,k_0)\,u^2 - \frac{1}{2}m\,u\,v + \frac{1}{2}k_0\,v^2\right)\right)\right) = 0\,, \qquad (16)$$

and

$$\frac{4m\,G_1(u,v)\left(-\frac{7}{2}u^3+\left(\left(-\frac{1}{4}+a\right)v+\frac{1}{2}w\right)u^2-\frac{3}{4}\,\left(v-\frac{1}{3}w\right)u\,v+\frac{1}{4}(av+w)\,v^2\right)}{(4u^2+v^2)(u^2-w^2-v\,w)} = 0, \qquad (17)$$

from equation (17), if $G_1 = 0$ then $f_n(u,v,w) = 0$, this gives that system (1) has no invariant algebraic surfaces. Or, if $m = 0$, this gives $k_2 = k_3 = 0$. Then equation (12) becomes

$$-w\frac{\partial}{\partial u} f(u,v,w) + (-u-w)\frac{\partial}{\partial v} f(u,v,w) + (3\,u - av + u^2 -\ w^2 - v\,w + b)\frac{\partial}{\partial w} f(u,v,w) = k_0\,f(u,v,w). \qquad (18)$$

It is not essay to discover a solution of equation (18). So, the weight change of variables is used as described in (Libre & Pessoa, 2009) in order to find an invariant algebraic surfaces of system (1). Let $u = \mu U, v = V,\quad w = W$ and $t = \mu T$, with $\mu \in \mathbb{C} \setminus \{0\}$. Then, system (1) turn into

$$\begin{aligned}\dot U &= -W\\ \dot V &= -\mu^2 U - \mu W\\ \dot W &= \mu^3 U^2 + 3\,\mu^2 U - a\,\mu V - \mu W^2 - \mu\,V\,W + b\mu,\end{aligned} \qquad (19)$$

where the dots denote the derivative of the variables U, V and W with respect to T. Set $F(U,V,W) = \mu^n f(\mu U, V, W) = \sum_{j=0}^{n} \mu^j F_j(U,V,W)$, where F_j is the weight homogeneous part with weight degree $n - j$ of F, and n is the weight degree of F with weight exponent $s = (1,0,0)$. And $K(U,V,W) = k_0$. Then, by invariant algebraic surfaces, we have

$$-W\sum_{j=0}^{n}\mu^j\ \frac{\partial}{\partial U}F_j(U,V,W) + (-\mu^2 U - \mu W\,)\sum_{j=0}^{n}\mu^j\ \frac{\partial}{\partial V}F_j(U,V,W) + (\,\mu^3 U^2 + 3\,\mu^2 U - a\,\mu V - \mu W^2 - \mu\,V\,W + b\mu)\sum_{j=0}^{n}\mu^j\ \frac{\partial}{\partial W}F_j(U,V,W) = k_0\ \sum_{j=0}^{n}\ F_j(U,V,W). \qquad (20)$$

We calculate the terms which contain μ^0 to obtain

$$-\frac{\partial}{\partial U}F_0(U,V,W)Z - k_0\,F_0(U,V,W) = 0,$$

that is

$$F_0(U,V,W) = G_0(V,W)\,e^{\frac{-k_0\,U}{W}},$$

where G_0 is a polynomial function of variables V and W. Since, $F_0(U,V,W)$ is a polynomial function. Thus, we obtain $k_0 = 0$. This implies that system (1) has no invariant algebraic surfaces with non-zero cofactors.

Theorem 3.3. System (1) has no rational first integrals.

Proof. From Proposition 3.2, system (1) has no Darboux polynomials. Then by Proposition 2.3, system (1) has no proper rational first integral.

We proved that in Proposition 3.2, system (1) does not have invariant algebraic surfaces. So, by Proposition 2.4, an exponential function must be in the following

$$E = e^{g(u,v,w)},$$

for more details see (Libre & Valls, 2012).

Proposition 3.4. System (1) has only two exponential factors e^u and e^v with cofactors $-w$ and $-u-w$, respectively.

Proof. Let $E = e^{g(u,v,w)}$, $g(u,v,w)=\sum_{k=0}^{n} g_k(u,v,w)$ be an exponential factor with non-zero cofactor $L = L_0 + L_1 u + L_2 v + L_3 w$, where each g_k is a homogeneous polynomial in its variables of degree k. Then, we have

$$-w \frac{\partial}{\partial u} e^{g(u,v,w)} + (-u-w)\frac{\partial}{\partial v} e^{g(u,v,w)} + (3\,u - a\,v + u^2 - w^2 - v\,w + b)\frac{\partial}{\partial w} e^{g(u,v,w)} = L\,e^{g(u,v,w)}. \tag{21}$$

Simplifying

$$-w \frac{\partial}{\partial u} g(u,v,w) + (-u-w)\frac{\partial}{\partial v} g(u,v,w) + (3\,u - a\,v + u^2 - w^2 - v\,w + b)\frac{\partial}{\partial w} g(u,v,w) = L. \tag{22}$$

Firstly, we assume that $n > 1$. calculating the terms of degree $n+1$ in equation (22), we take out

$$(u^2 - w^2 - v\,w)\frac{\partial}{\partial w} g_n(u,v,w) = 0,$$

that is

$$g_n(u,v,w) = F_1(u,v),$$

where F_1 is a polynomial of degree n. Also, computing the terms of degree n in equation (22), we obtain

$$-w \frac{\partial}{\partial u} F_1(u,v) + (-u-w)\frac{\partial}{\partial v} F_1(u,v) + (3\,u - a\,v)\frac{\partial}{\partial w} F_1(u,v) + (u^2 - w^2 - v\,w)\frac{\partial}{\partial w} g_{n-1}(u,v,w) = 0,$$

this gives

$$g_{n-1}(u,v,w) = \frac{1}{\sqrt{4u^2+v^2}}\left(\left((2u-v)\left(\frac{\partial}{\partial v}F_1(u,v)\right) - v\left(\frac{\partial}{\partial u}F_1(u,v)\right)\right)\operatorname{arctanh}\left(\frac{v+2w}{\sqrt{4u^2+v^2}}\right) + \left(\left(-\frac{1}{2}\frac{\partial}{\partial u}F_1(u,v) - \frac{1}{2}\frac{\partial}{\partial v}F_1(u,v)\right)\ln(-u^2 + w^2 + v\,w) + F_2(u,v)\right)\sqrt{4u^2+v^2}\right)$$

Since $g_{n-1}(u,v,w)$ is a polynomial of degree $n-1$, then we have

$$\frac{\partial}{\partial u}F_1(u,v) + \frac{1}{2}\frac{\partial}{\partial v}F_1(u,v) = 0 \tag{23}$$

and

$$(2u-v)\left(\frac{\partial}{\partial v}F_1(u,v)\right) - v\left(\frac{\partial}{\partial u}F_1(u,v)\right) = 0, \tag{24}$$

it is clearly that the solution of equation(23) is

$$F_1(u,v) = F_3(v-u),$$

where F_3 is a polynomial of the variables u and v. Since, $F_1(u,v)$ is a polynomial of degree n, then it must be in the formula

$$F_1(u,v) = c\,(v-u)^n, \tag{25}$$

where c is arbitrary constant. Putting (25) in (24) we take out

$$c\,n\,(v-u)^n = 0.$$

Since, $n > 1$ then $c = 0$, this implies that $F_1 = 0$, this gives $g_n = 0$. Thus $g = 0$, for $n > 1$. Now, we assume that $g(u,v,w)$ is a polynomial of degree $n = 1$.

Letting $g(u,v,w) = c_0 + c_1 u + c_2 v + c_3 w$.

Then, by equation (22), we have

$$-w\,c_1 + (-u-w)\,c_2 + (3\,u - a\,v + u^2 - w^2 - v\,w + b)\,c_3 = L_0 + L_1 u + L_2 v + L_3 w.$$

Comparing the coefficient, we obtain $c_3 = L_2 = L_0 = 0$, $c_1 = L_1 - L_3$ and $c_2 = -L_1$.

That is

$$g(u,v,w) = (L_1 - L_3)\,u - L_1\,v.$$

This implies that $e^{(L_1-L_3)\,u - L_1\,v}$ is the exponential factor with cofactor $L_1 u + L_3 w$. Hence, the only two independent exponential factors of system (1) are e^u and e^v with cofactors $-w$ and $-u-w$, respectively.

Now, having a Darboux first integral is illustrated in the following theorems.

Theorem 3.5. System (1) has no Darboux first integrals.

Proof. Since, e^u and e^v are the unique exponential factors with cofactors $-w$ and $-u-w$, respectively. Then by Darboux Theorem 2.5, we have

$$\mu_1(-w)+\mu_2(-u-w)=0, \qquad (26)$$

with non-zero constants μ_1, $\mu_2 \in \mathbb{C}$. The above equation has no non-trivial solution. Then, system (1) does not have a Darboux first integral.

The condition $ab \neq 0$ that has assumed in system (1) is an essential condition to prove the existence of C^1 and an analytic first integral.

Theorem 3.6. If $a \in (0,3), b>0$ and $3b-ab>a^2$ then system (1) has no a global C^1 first integral.

Proof. Since $s_0=\left(0,\frac{b}{a},0\right)$ is the equilibrium point of system (1), then the Jacobian matrix at s_0 of system (1) is

$$J=\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 3 & -a & -\frac{b}{a} \end{bmatrix}.$$

A characteristic equation of the above matrix is

$$P(\lambda)=\lambda^3+\frac{b}{a}\lambda^2+(3-a)\,\lambda+a=0.$$

The eigenvalues are

$$\lambda_1=\frac{A^{\frac{1}{3}}}{6\,a}+\frac{6\,a^3-18a^2+2b^2}{3a\,A^{\frac{1}{3}}}-\frac{b}{3a} \quad \text{and}$$

$$\lambda_{2,3}=-\frac{A^{\frac{1}{3}}}{12\,a}-\frac{3\,a^3-9a^2+b^2}{3a\,A^{\frac{1}{3}}}-\frac{b}{3a}\pm\frac{\sqrt{3}\,i}{2}\left(\frac{A^{\frac{1}{3}}}{6a}-\frac{6\,a^3-18a^2+2b^2}{3a\,A^{\frac{1}{3}}}\right),$$

where

$$A=-108a^4-36\,a^3b+12\,a^2\sqrt{-12a^5+189a^4+54a^3b-3a^2b^2-324a^3-162a^2b+18ab^2+12b^3+324a^2-27b^2}+108a^2b-8b^3.$$

Then by Theorem 2.1 the eigenvalues have non-zero negative real parts if and only if $a \in (0,3), b>0$ and $3b-ab>a^2$. Then, by Theorem 2.2 system (1) has no global C^1 first integrals in the neighborhood of s_0.

Proposition 3.7. The linear part of system (1) has no polynomial first integrals at the equilibrium point $s_0=\left(0,\frac{b}{a},0\right)$, where a and b satisfy

$$27a^4-b^3=0,$$
$$3a^3-9a^2+b^2=0. \qquad (27)$$

Proof. Firstly, we use the linear transformation $(u,v,w)\to\left(u,v+\frac{b}{a},w\right)$ to move s_0 into the origin by then system (1) becomes

$$\dot{u}=-w,$$

$$\dot{v}=-u-w,$$

$$\dot{w}=3\,u-a\,v-\frac{b}{a}\,w-v\,w+u^2-w^2. \qquad (28)$$

The linear part can be written of the above system as

$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix}=\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 3 & -a & -\frac{b}{a} \end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}. \qquad (29)$$

The characteristic equation at $(0,0,0)$ is

$$u^3+\frac{b}{a}u^2+(3-a)\,u+a=0. \qquad (30)$$

Simply, we can see that equation (30) has a triple real root say λ if and only if it could be written as

$$(u-\lambda)^3=u^3-3\,\lambda\,u^2+3\,\lambda^2u-\lambda^3.$$

That is, $\lambda=\frac{-b}{3\,a}$, with

$$L_1(a,b)=\lambda^3+a, \qquad L_2(a,b)=3-a-3\lambda^2.$$

Putting the value of $\lambda=-\frac{b}{3\,a}$ in L_1 and L_2. We compute the value of a and b such that $L_1=L_2=0$, to obtain

$$27a^4-b^3=0,$$
$$3a^3-9a^2+b^2=0.$$

This implies that $\lambda_1 = \lambda_2 = \lambda_3 = -\frac{b}{3\,a}$, then from cases 1-3 in Proposition 2.6, we obtain the linear part of system (1) has no a polynomial first integrals.

Theorem 3.8. If a and b satisfy the condition (27), then system (1) does not have analytic first integrals at $s_0 = \left(0, \frac{b}{a}, 0\right)$.

Proof. Firstly, we move the equilibrium point $\left(0, \frac{b}{a}, 0\right)$ into the $(0,0,0)$. Then, by the linear change of coordinates $(u, v, w) \to \left(u, v + \frac{b}{a}, w\right)$, system (1) can be transformed into system (28). Suppose that $H = \sum_{i \geq 1} H_i(x, y, z)$ is analytic first integral of system (1), where H_i is a homogeneous polynomial of degree i for all $i \geq 1$. We will illustrate by induction that

$$H_i = 0 \quad \text{for all } i \geq 1.$$

Since, H is a first integral of system (28), then by definition of first integral, we have

$$-w\,\frac{\partial H}{\partial u} + (-u - w)\frac{\partial H}{\partial v} + \left(3\,u - a\,v - \frac{b}{a}\,w - v\,w + u^2 - w^2\right)\frac{\partial H}{\partial w} = 0. \qquad (31)$$

Calculating the terms of degree 1 in equation (31), we take out

$$-w\,\frac{\partial}{\partial u}\,H_1(u, v, w) + (-u - w)\frac{\partial}{\partial v}H_1(u, v, w) + \left(3\,u - a\,v - \frac{b}{a}\,w\right)\frac{\partial}{\partial w}H_1(u, v, w) = 0. \qquad (32)$$

Then H_1 could be a zero polynomial or it could be a polynomial first integral of first degree. Since, a and b satisfy the condition (27), then, by Proposition 3.7, the linear part of system (28) has no polynomial first integral. This gives $H_1 = 0$, which proves $H_i = 0$ for $i = 1$.

Now, assume that $H_i = 0$ for $i = 1, \dots, m-1$ with $m \geq 2$, and it will be proved for $i = m$. Using induction supposition, calculating the terms of degree m in equation (31), we obtain

$$-w\,\frac{\partial}{\partial u}\,H_m(u, v, w) + (-u - w)\frac{\partial}{\partial v}\,H_m(u, v, w) + \left(3\,u - a\,v - \frac{b}{a}\,w\right)\frac{\partial}{\partial w}\,H_m(u, v, w) = 0. \qquad (33)$$

Then, H_m could be a zero polynomial or it could be a polynomial first integral of m degree. Since, a and b satisfy the condition (27), we proceed as the case H_i for $i = 1$, and using by Proposition 3.7, we obtain that $H_m = 0$. This proves that $H_i = 0$ for all $i \geq 1$. Thus, system (28) has no analytic first integral. Going back under the change of coordinates $(u, v, w) \to \left(u, v + \frac{b}{a}, w\right)$, gives that system (1) does not have an analytic first integral at s_0.

4. CONCLUSION

In this paper, we proved that the Kingni–Jafari system has no Darboux first integrals. Also, this system has no analytic first integrals at the neighborhood of the equilibrium point and we obtained that the system has no global C^1 first integrals for $a \in (0,3), b > 0$ and $3b - ab > a^2$.

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