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RESEARCH PAPER

Centre Bifurcations for a Three Dimensional System with Quadratic Terms

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A B S T R A C T:

This article is devoted to study the bifurcated periodic orbits from centre for a differential equation of third order. Sufficient conditions for the existence of a centre are obtained by using inverse Jacobi multiplier. As a result, we found four sets of centre conditions on the centre manifold. For a given centre, it is shown that three periodic orbits can be bifurcated from the origin under two sets of condition and four periodic orbits under the other sets of condition. The cyclicityes are obtained by considering the linear parts of the corresponding Liapunov quantities of the perturbed system.

KEY WORDS: Hopf and Centre Bifurcation; Periodic Solutions; Inverse Jacobi Multiplier.
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INTRODUCTION

We consider the following third order differential equation

$$\dddot{x} - \alpha\,\ddot{x} - \beta\,\dot{x} - \gamma\,x - H(x\,,\dot{x},\ddot{x}) = 0, \qquad (1)$$

where $\alpha\,,\beta\,,\gamma\ \in\ \mathbb{R}$ and $H(x\,,\dot{x},\ddot{x})$ is an analytic quadratic function. When $\alpha = \beta = \gamma = -1$, the centre problem on a local centre manifold of equation (1) is studied in (Mahdi, 2013). By eliminating two coefficients of the quadratic function H, he has found the necessary and conditions for the existence of a centre on the centre manifold for the three 4-parameter families of equation (1). Mahdi et al. (2017) have constructed a hybrid approach using numerical algebraic geometry to the center-focus problem. They applied their technique to have centre conditions for equation (1) (Mahdi, et al., 2017).

Equation (1) can be transformed into a system of nonlinear equations. This can be introducing $\dot{x} = y,\ \ddot{x} = z$ to obtain

$$\begin{aligned} \dot{x} &= y && = P_1(x,y,z), \\ \dot{y} &= z && = P_2(x,y,z), \qquad (2) \\ \dot{z} &= \alpha\,z + \beta\,y + \gamma\,x + H(x\,,y\,,z) && = P_3(x,y,z), \end{aligned}$$

where $H(x,y,z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4\,xy + a_5xz + a_6yz$. System above has an isolated critical point at the origin, the Jacobian matrix of system (2) at that point has a zero and

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two pure imaginary eigenvalues under some conditions on the parameters. In that case, the origin is called zero-Hopf critical point (for more information see (Salih, 2017)). Furthermore, under some other conditions on the parameters, the Jacobian matrix of system (2) at the origin has a non-zero and a pair of pure imaginary eigenvalues, in such case the origin is called Hopf point. For the three dimensional systems (2), a sufficient condition for a Hopf bifurcation is explained below. The characteristic polynomial for system (2) is given by

$$\lambda^3 - T\,\lambda^2 - K\,\lambda - D = 0, \qquad (3)$$

such that

i. $T = \sum_{i=1}^{3} b_{i,i}$ (trace of the Jacobian matrix of system (2) at the origin),
ii. $D =$ determinant of the Jacobian matrix of system (2) at the origin,
iii. $K = -(B_1 + B_2 + B_3)$,

where $B_1 = b_{2,2}b_{3,3} - b_{2,3}b_{3,2}$, $B_2 = b_{1,1}b_{3,3} - b_{1,3}b_{3,1}$, $B_3 = b_{2,2}b_{1,1} - b_{2,1}b_{1,2}$ and $b_{i,j}$, $i,j = 1,2,3$ are elements of the Jacobian matrix of system (2) at the origin (for more detail on Hopf bifurcation also reader can consult (Ameen, et al., 2009), (Salih, 2009) and (Salih & Ameen, 2008)). Then the Hopf bifurcation take place at a point, Hopf point, on the surface

$$TK + D = 0\,; \quad K < 0 \quad \text{and} \quad T \neq 0\,. \qquad (4)$$

Using the following change of variables with Hopf conditions

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & \frac{1}{\alpha^2} \\ \omega & 0 & \frac{1}{\alpha} \\ 0 & -\omega^2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \qquad (5)$$

we can write system (2) as

$$\begin{aligned} \dot{y}_1 &= -\omega y_2 + G(y_1\,, y_2\,, y_3), \\ \dot{y}_2 &= \omega y_1 + \frac{\omega}{\alpha} G(y_1\,, y_2\,, y_3), \\ \dot{y}_3 &= \alpha y_3 - \alpha\omega\, G(y_1\,, y_2\,, y_3), \end{aligned} \qquad (6)$$

where $\omega = \sqrt{-\beta}$ and

$$G(y_1, y_2, y_3) = -\frac{\alpha\omega a_2 {y_1}^2}{\omega^2+\alpha^2} + \frac{(\omega^2 a_6 - a_4)\alpha\, y_1 y_2}{\omega^2+\alpha^2} - \frac{(\alpha^2 a_6 + 2\alpha a_2 + a_4) y_1 y_3}{\alpha(\omega^2+\alpha^2)} + \frac{1}{\alpha\,\omega(\omega^2+\alpha^2)}(-\alpha^2(\omega^4 a_3 - \omega^2 a_5 + a_1)\, {y_2}^2 + (2\alpha^2\omega^2 a_3 + \alpha\omega^2 a_6 + \omega^2 a_5 - \alpha^2 a_5 - \alpha a_4 - 2a_1) y_3 y_2 - \frac{1}{\alpha^3\omega(\omega^2+\alpha^2)}(\alpha^4 a_3 + \alpha^3 a_6 + \alpha^2 a_2 + \alpha^2 a_5 + \alpha a_4 + a_1) y_3^2.$$

There are two methods to solve the centre problems of system (6) at the Hopf point. The classical method which is called Lyapunov Centre Theorem (for more detail see (Bibikov, 1979)) and the inverse Jacobi multiplier is the modern method (for more detail on this method see (Berrone & Giacomini, 2003) and (Buica, et al., 2012)). The nonzero smooth function V is called inverse Jacobi multiplier of system (6), if it satisfies the following partial differential equation:

$$\chi(V) = V\,div(\chi), \qquad (7)$$

where χ is a vector field associated to (6) and div refers to the divergence operator. Using inverse Jacobi multiplier, Buică, et al. (2012) solves the centre problem by the following theorem.

Theorem 1. System (6) has a centre at the origin if and only if it admits a local analytic inverse Jacobi multiplier of the form $V(y_1, y_2, y_3) = y_3 + \cdots$ in a neighborhood of the origin in $\mathbb{R}^3$. Moreover, when such V exists, the local analytic centre manifold, W^c, lies in $V^{-1}(0)$.

Remark 1. The Hopf critical point $u^* \in \mathbb{R}^3$ is a centre of system (2) if and only if there is an inverse Jacobi multiplier V at the Hopf point where $\nabla V(u^*) \neq 0$.

Mahdi (2013) has studied the center problem of system (2) which has quadratic nonlinearities. For the existence of a center, the necessary and sufficient conditions were found. In this article, the inverse Jacobi multiplier is used to find centre conditions on the centre manifold of system (2). Then, we perturbed the parameters to obtain a number of bifurcated periodic orbits.

The layout of the article is as follows. The sufficient conditions for the existence of a centre are studied in section one. The summery of the cyclicity technique is presented in section two. Section Three is devoted to apply the cyclicity technique to find number of periodic orbits bifurcating from centre for the third order differential equation. The conclusions are finally made. Throughout this paper, MAPLE software is used to verify calculations and also to plot figures.

1. CENTRE CONDITIONS

The primary purpose of this section is to present sufficient conditions for the existence of the Hopf bifurcation and the centre on the centre manifold in the three dimensional system (2). The Hopf critical point is called centre if there exists a neighborhood U of the point such that all orbits are periodic on it. Furthermore, if all the orbits have the same period, it is called isochronous center (Ameen, 2015).

Proposition 1. System (2) has a Hopf point at the origin if and only if the following conditions are satisfied:

$$\gamma = -\alpha\,\beta \;,\; \alpha \neq 0 \text{ and } \beta < 0\,. \qquad (8)$$

Proof: First, we shall prove the necessary conditions (8) and let the origin be a Hopf point. The characteristic equation of the Jacobian matrix of system (2) at the origin is given by

$$\lambda^3 - \alpha\,\lambda^2 - \beta\,\lambda - \gamma = 0\,. \qquad (9)$$

If we compare the equation above with equation (3), the following values of T, K and D are obtained:

$$T = \alpha\,,\; K = \beta \text{ and } D = \gamma. \qquad (10)$$

Since the origin is a Hopf point, then the parameters in equation (10) satisfy equation (4):

$$TK + D = 0 \;\Rightarrow\; \alpha\beta + \gamma = 0 \;\Rightarrow\; \gamma = -\alpha\beta,$$
$$T \neq 0 \text{ and } K < 0 \;\Rightarrow\; \alpha \neq 0 \text{ and } \beta < 0\,.$$

Therefore, the conditions are held.

Conversely: We shall prove sufficiency. Assume that $\gamma = -\alpha\beta$, $\alpha \neq 0$ and $\beta < 0$. Since $\beta < 0$, we can assume that $\beta = -\omega^2$. The Jacobian matrix of system (2) at the origin becomes

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & -\omega^2 & \alpha \end{bmatrix}, \qquad (11)$$

and its eigenvalues are $\lambda_{1,2} = \pm\omega i$ and $\lambda_3 = \alpha$. This means that the Jacobian matrix of system (2) at the origin has a pair of purely imaginary and a nonzero eigenvalues. Thus, the origin is a Hopf point. □

Now, we are looking for the inverse Jacobi multiplier function for system (2). The vector field of the system is denoted by χ :

$$\chi = \big(P_1(x,y,z), P_2(x,y,z), P_3(x,y,z)\big) \qquad (12)$$

which is a quadratic vector field and we let V be an inverse Jacobi multiplier for system (2), which is defined by

$$V = \sum_{k=0}^{2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{(k-j,j-i,i)}\, x^{k-j}\, y^{j-i}\, z^{i}$$
$$= C_{2,0,0}x^2 + C_{1,1,0}\,xy + C_{1,0,1}\,xz + C_{0,2,0}y^2 + C_{0,1,1}\,yz$$
$$+ C_{0,0,2}z^2 + C_{1,0,0}x + C_{0,1,0}y + C_{0,0,1}z + C_{0,0,0}, \qquad (13)$$

where $C_{i,j,k} \in \mathbb{R}\,, i,j,k = 0,1,2$.

Proposition 2. System (2) has an inverse Jacobi multiplier if and only if one of the following conditions are satisfied:

$i.\, \tau_1 = \{a_1 = \omega^2\alpha(\omega^2 - 1), a_2 = a_3 = \alpha, a_4 = a_6 = 0, a_5 = 2\alpha(\omega^2 - 1)\},$

$ii.\, \tau_2 = \{a_1 = a_5\omega^2, a_2 = a_3 = 0, a_4 = \omega^2 a_6\},$

$iii.\, \tau_3 = \left\{a_1 = \frac{-1}{2}\alpha a_4\,, a_2 = a_3 = a_5 = a_6 = 0\right\},$

$iv. \tau_4 = \{a_1 = -\alpha\omega^2, a_2 = \alpha, a_5 = -2\alpha,\ a_3 = a_4 = a_6 = 0\}$,
where $\omega = \sqrt{-\beta}$.

Proof. First, we shall prove that the conditions τ_1, τ_2, τ_3 and τ_4 are necessary. Assume that system (2) has an inverse Jacobi multiplier, V, which is defined in (13). Then it satisfies the following partial differential equation

$$\chi(V) = V\,\mathrm{div}(\chi),$$

where χ is a vector field of (2) which is defined in (12):

$$\chi(V) = \frac{\partial V}{\partial x} P_1(x,y,z) + \frac{\partial V}{\partial y} P_2(x,y,z) + \frac{\partial V}{\partial z} P_3(x,y,z) = (C_{1,0,1}z + C_{1,1,0}y + 2C_{2,0,0}x + C_{1,0,0})y + (C_{0,1,1}z + 2C_{0,2,0}y + C_{1,1,0}x + C_{0,1,0})z \ + (2C_{0,0,2}z + C_{0,1,1}y + C_{1,0,1}x, + C_{0,0,1})(\alpha\, z + \ \beta\, y + \gamma\, x + H(x\ , y\ , z))$$

and

$$\mathrm{div}\,(\chi) = \frac{\partial P_1(x,y,z)}{\partial x} + \frac{\partial P_2(x,y,z)}{\partial y} + \frac{\partial P_3(x,y,z)}{\partial z} = \alpha + a_5 x + a_6 y + 2a_3 z\ .$$

After solving $\chi(V) - \mathrm{div}(\chi)V = 0$, the set of solutions τ_1, τ_2, τ_3 and τ_4 can be obtained.

Conversely: We shall now prove sufficiency. Assume that condition τ_1 holds. Thus, we consider the vector field of system (2)

$$\chi = (y, z, \alpha\,\omega^2 x - \omega^2 y + \alpha\, z + \omega^2\alpha\,(\omega^2 - 1)x^2 + 2\alpha\,(\omega^2 - 1)\,xz + \alpha\, y^2 + \alpha\, z^2).$$

From

$$\chi(V) = V\,\mathrm{div}(\chi)\ ,$$

where

$$V = \sum_{k=0}^{2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{(k-j,j-i,i)}\, x^{k-j}\, y^{j-i}\, z^{i}$$

and $\mathrm{div}\chi = 2\alpha(\omega^2 - 1)x + 2\alpha\, z + \alpha$ is divergent of the vector field χ, the following function is obtained

$$V = \omega^2 x + z + (\omega^4 - \omega^2)x^2 + 2(\omega^2 - 1)xz + y^2 + z^2$$

which is the inverse Jacobi multiplier of system (2).

Assume condition τ_2 holds. The vector field of system (2) becomes:

$$\chi = (y, z, \alpha\omega^2 x - \omega^2 y + \alpha z + \omega^2 a_5 x^2 + \omega^2 a_6 xy + a_5 xz + a_6 yz)$$

from $\chi(V) = V\,\mathrm{div}\chi$, where V defined above and $\mathrm{div}\chi = a_5 x + a_6 y + \alpha$ is divergent of χ, the following inverse Jacobi multiplier is obtained

$$V = \omega^2 x + z. \tag{14}$$

Assume condition τ_3 holds. Then the vector field of system (2) is given by

$$\chi = (y, z, (\alpha z - \omega^2 y + \alpha\omega^2 x - \tfrac{1}{2}\alpha a_4 x + a_4 xy))$$

from $\chi(V) = V\,\mathrm{div}\chi$ where $div\chi = \alpha$ and V defined above, we obtain the following inverse Jacobi multiplier for system (2)

$$V = \omega^2 x + z - \tfrac{1}{2}a_4 x^2.$$

Assume condition τ_4 holds. Then, we consider the vector field of system (2)

$$\chi = (y, z, \alpha\omega^2 x - \omega^2 y + \alpha z - \alpha\,\omega^2 x^2 + \alpha\, y^2 - 2\,\alpha\, xz)$$

from $\chi(V) = V\,\mathrm{div}\chi$ where $\mathrm{div}\chi = \alpha - 2\alpha x$ and V defined above, the following inverse Jacobi multiplier is obtained

$$V = \omega^2 x + z - \omega^2 x^2 - 2xz + y^2.$$

□

The inverse Jacobi multiplier is used to find sufficient conditions for a critical point to be a centre for the three dimensional system (2). The explicit formula for the inverse Jacobi multiplier for system (2) is given by the following propositions.

Proposition 3. The Hopf critical point at the origin is center of system (2) if the parameters satisfy the following conditions

$i. \tau_1 = \{a_1 = \omega^2\alpha(\omega^2 - 1), a_2 = a_3 = \alpha, a_4 = a_6 = 0, a_5 = 2\alpha(\omega^2 - 1)\},$

$ii. \tau_2 = \{a_1 = a_5\omega^2, a_2 = a_3 = 0, a_4 = \omega^2 a_6\}$

$iii. \tau_3 = \{a_1 = \frac{-1}{2}\alpha a_4, a_2 = a_3 = a_5 = a_6 = 0\}$

$iv. \tau_4 = \{a_1 = -\alpha\omega^2, a_2 = \alpha, a_5 = -2\alpha, a_3 = a_4 = a_6 = 0\}.$

Proof. It is easy to prove the above proposition by finding the inverse Jacobi multiplier corresponding to each set of conditions (Buica, et al., 2012). The inverse Jacobi multiplier corresponding to each set of conditions and passing through the origin are

- $V = \omega^2 x + z + (\omega^4 - \omega^2)x^2 + 2(\omega^2 - 1)xz + y^2 + z^2$ Corresponding to τ_1 (see Figure 1 (a)),
- $V = \omega^2 x + z$ Corresponding to τ_2 (see Figure 2 (a)),
- $V = \omega^2 x + z - \frac{1}{2}a_4 x^2$ Corresponding to τ_3 (see Figure 2 (b)),
- $V = \omega^2 x + z - \omega^2 x^2 - 2xz + y^2$ Corresponding to τ_4 (see Figure 1 (b)).

Since $\nabla V(0,0,0) = \omega^2 + 1 \neq 0$ for each case, then Theorem 1 indicates that the Hopf critical point at the origin is a centre.

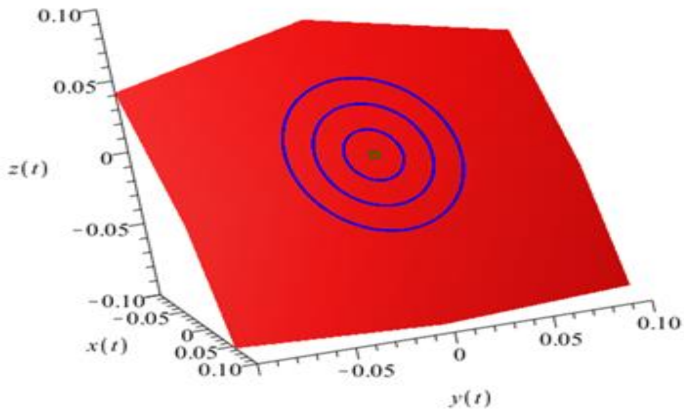
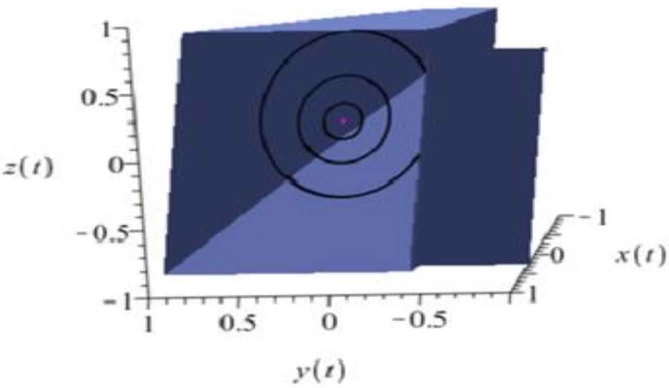


Figure 1. (a): Phase portrait of system (2) satisfying conditions τ_1 , $\omega = a_3 = 1$, with initial points (0.01, 0.01, -0.0102), (0.02, 0.02, -0.0208), (0.03, 0. 03, -0.0318). The green point is the critical point and the red plane is the inverse Jacobi multiplier $V(x, y, z) = (\omega^4 - \omega^2)x^2 + 2(\omega^2 - 1)xz + y^2 + z^2 + \omega^2 x + z$.



(b) Phase portrait of system (2) satisfying conditions τ_3 , $\omega = a_4 = \alpha = 1$, with initial points (0.01, 0.01, -0.00995), (0.02, 0.02, -0.0198), (0.03, 0. 03, -0.02955). The magenta point is the critical point and the Niagara Azure plane is the inverse Jacobi multiplier $V(x, y, z) = -\frac{1}{2}a_4 x^2 + \omega^2 x + z$.

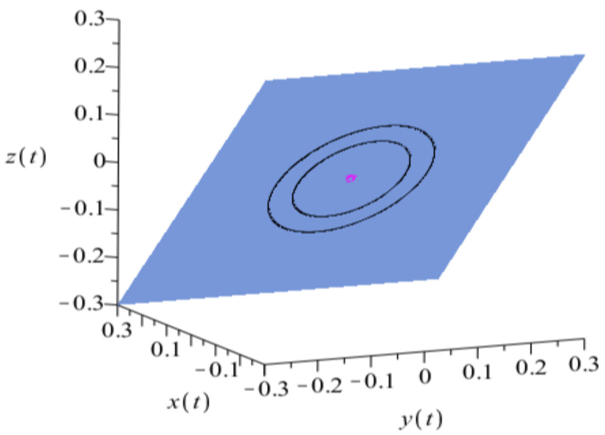
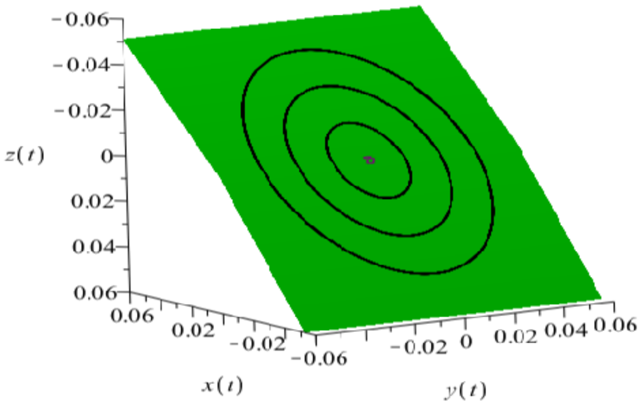


Figure 2. (a): Phase portrait of system (2) satisfying conditions τ_2 , $\omega = a_5 = a_6 = \alpha = 1$, with initial points (0, 0.1, 0), (0.1, 0.1, -0.1). The magenta point is the critical point and the Niagara Azure plane is the inverse Jacobi multiplier $V(x, y, z) = \omega^2 x + z$.



(b) Phase portrait of system (2) satisfying conditions τ_4 , $\omega = a_5 = 1$, with initial points (0., 0.1, 0), (0.1, 0.2, -0.125), 0.2, 0. 3, -0.3333333333). The magenta point is the critical point and the Niagara Azure plane is the inverse Jacobi multiplier $V(x, y, z) = -\omega^2 x^2 + y^2 - 2xz + \omega^2 x + z$.

2. A CYCLICITY TECHNIQUE IN $\mathbb{R}^3$

In the bifurcation theory, one of the contemporary research areas is the bifurcation of limit cycles from centre. They are obtained by perturbing a focus or centre. Here, we consider system (6), the set of all parameters in $G(y_1, y_2, y_3)$ and the corresponding parameter space is denoted by Λ and $\boldsymbol{K}$, respectively.

In two dimensional systems, for the first time Christopher (2005) has explored a useful technique for examining the cyclicity bifurcating from the centre by linearizing the Liapunov quantities (Christopher, 2005). Salih (2015) has generalized the technique to three dimensional systems to examine the cyclicity bifurcating from centres (Salih, 2015). In addition, Salih and Hasso, have used the same technique to study the bifurcated periodic orbits in a three dimensional system and the L$\ddot{u}$ system (Salih & Hasso, 2017). We summarize the technique which is used to estimate the cyclicity in three dimensional system as follows.

1. A point will be selected for a centre variety.
2. We linearize the Liapunov quantities around this point.
3. We check the codimension of the point. If the codimension of the point be r where the first r linear terms of Liapunov quantities are linearly independent, then $r-1$ limit cycles can be bifurcated by small perturbation.

Composing the Liapunov function and finding its Liapunov quantities is a way to determine the number and stability of the limit cycles. In this method, we define a function of the form

$$F(x,y,z) = y_1^2 + y_2^2 + \sum_{k=3}^{\infty} F_k(y_1, y_2, y_3), \qquad (15)$$

where

$$F_k = \sum_{i=0}^{k} \sum_{j=0}^{j} C_{(k-i,i-j,j)} y_1^{k-i}\, y_2^{i-j} y_3^{j}.$$

For system (6) and the coefficients of F_k satisfy

$$\chi(F) = L_1(y_1^2 + y_2^2) + L_2(y_1^2 + y_2^2)^2 + L_3(y_1^2 + y_2^2)^3 + \cdots,$$

(16)

where $L_i\,, i = 1,2, \ldots$ are polynomials and the L_i is the i^{th} Liapunov constant. We assume that $0 \in K$ corresponds to the centre of system (6). Using a perturbation technique, the following are obtained

$$\chi = \chi_o + \chi_1 + \cdots,$$

$$F = F_o + F_1 + \cdots, \qquad (17)$$

$$L_i = L_{io} + L_{i1} + \cdots, \qquad i = 1,2, \ldots,$$

where χ_o, F_o and L_{io} are calculated at the unperturbed terms and χ_1, F_1 and L_{1i} are obtained at the perturbed terms of first order (the terms of degree one in Λ), and so forth. Both the Liapunov function F_i and the Liapunov quantity L_i have same degree in parameters which is i. Substituting equation (17) into equation (16), the following equations are obtained

$$\chi_o F_o = 0\,,$$

$$\chi_o F_1 + \chi_1 F_o = L_{11}(y_1^2 + y_2^2) + L_{21}(y_1^2 + y_2^2)^2 + \cdots \quad (18)$$

and more general,

$$\chi_o F_i + \cdots + \chi_i F_o = L_{1i}(y_1^2 + y_2^2) + L_{2i}(y_1^2 + y_2^2)^2 + \cdots \quad (19)$$

Solving the pair equations (18) simultaneously, the linear terms of the Liapunov quantities L_k (modulo the $L_i\,, i < k$) will be obtained. To obtain the higher order terms of the Liapunov quantities, equation (19) is used.

3. CENTRE BIFURCATION FOR SYSTEM (2)

In this section, the technique which is presented in the previous section is applied to examine the cyclicity bifurcating from the center at the origin of system (2) where the parameters fulfill the conditions in Proposition 3. The main outcome of this section are the theorems below.

Theorem 2. Four limit cycles can bifurcate from the critical point at the origin when the parameters in system (2) satisfy conditions τ_1 or τ_4 of Proposition 3, proved that $\omega \geq 1$.

Proof. When condition τ_1 hold, system (2) reduces to

$$\dot{x} = y,$$
$$\dot{y} = z, \qquad (20)$$
$$\dot{z} = \alpha\omega^2 x - \omega^2 y + \alpha z + \omega^2\alpha(\omega^2-1)x^2 + \alpha y^2 + \alpha z^2 + 2\alpha(\omega^2-1)xz,$$

and the transformed system (6) is obtained where

$$G(y_1,y_2,y_3) = -\frac{\alpha^2\omega {y_1}^2}{\alpha^2+\omega^2} - 2\frac{\alpha y_1 y_3}{\alpha^2+\omega^2} - \frac{\alpha^2\omega {y_2}^2}{\alpha^2+\omega^2} + 2\frac{\alpha^2 y_2 y_3}{\omega(\alpha^2+\omega^2)} - \frac{(\alpha^2+\omega^2-1){y_3}^2}{\alpha^2\omega} \quad . \qquad (21)$$

It is easy to define the Liapunov function of F_o of equation (6) which it satisfies $\chi_o F_o = 0$:

$$\chi_o = \left(-\omega y_2 + G(y_1,y_2,y_3), \omega y_1 + \tfrac{\omega}{\alpha}G(y_1,y_2,y_3), \alpha y_3 - \alpha\omega G(y_1,y_2,y_3)\right),$$

where $G(y_1,y_2,y_3)$ is defined in (21) and

$$F_o = {y_1}^2 + {y_2}^2 + \sum_{k=3}^{N}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{(k-j,j-i,i)}\, {y_1}^{k-j}{y_2}^{j-i}{y_3}^{i}$$

We choose a point from center variety, $(a_1,a_2,a_3,a_4,a_5,a_6,\alpha,\beta,\gamma) = (\omega^2(\omega^2-1),1,1,0,2(\omega^2-1),0,1,-\omega^2,\omega^2)$ and we let $a_1 = \omega^2(\omega^2-1)+\epsilon_1$, $a_2 = 1+\epsilon_2$, $a_3 = 1+\epsilon_3$, $a_4 = 0+\epsilon_4$, $a_5 = 2(\omega^2-1)+\epsilon_5$, $a_6 = 0+\epsilon_6$, $\alpha = 1+\epsilon_7$, $\beta = -\omega^2+\epsilon_8$, $\gamma = \omega^2+\epsilon_9$, where $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6,\epsilon_7,\epsilon_8$ and ϵ_9 are parameters after perturbation in the system.

Transformation (5) is also used for perturbed part of vector field of system (6). Using MAPLE software and solving equation (18), the following linear independent terms of Liapunov quantities are obtained:

1. $L_1 = \frac{1}{\omega^2+1}(\omega^2\epsilon_7 - \epsilon_8 - \epsilon_9)$

2. $L_2 = \frac{-1}{4\omega^6+9\omega^4+6\omega^2+1}(3\omega^6\epsilon_3 + 4\omega^6\epsilon_5 + 2\omega^6\epsilon_6 - 16\omega^6\epsilon_7 - 8\omega^6\epsilon_8 - 8\omega^4\epsilon_1 - 3\omega^4\epsilon_2 + 3\omega^4\epsilon_3 - 6\omega^4\epsilon_4 + 2\omega^4\epsilon_5 + \omega^4\epsilon_6 + 26\omega^4\epsilon_7 + 14\omega^4\epsilon_8 + 16\omega^4\epsilon_9 - 7\omega^2\epsilon_1 - 3\omega^2\epsilon_2 - 6\omega^2\epsilon_4 - 2\omega^2\epsilon_5 - \omega^2\epsilon_6 + 9\omega^2\epsilon_7 - 21\omega^2\epsilon_8 - 26\omega^2\epsilon_9 + \epsilon_1 - 10\epsilon_8 - 9\epsilon_9)$,
3. $L_3 = \frac{1}{(16\omega^2+1)(9\omega^2+1)(\omega^2+4)(4\omega^2+1)^2(\omega^2+1)^4}((3456\epsilon_3 - 2304\epsilon_5 + 4608\epsilon_8)\omega^{20} + (4608\epsilon_1 - 3456\epsilon_2 + 13560\epsilon_3 - 2304\epsilon_4 - 28048\epsilon_5 - 8064\epsilon_6 + 32256\epsilon_7 + 44576\epsilon_8)\omega^{18} + (-32256\epsilon_9 + 52640\epsilon_1 - 13560\epsilon_2 + 994\epsilon_3 + 1904\epsilon_4 - 78976\epsilon_5 - 36344\epsilon_6 + 125408\epsilon_7 + 45936\epsilon_8)\omega^{16} + (-125408\epsilon_9 + 144392\epsilon_1 - 994\epsilon_2 + 36522\epsilon_3 + 43824\epsilon_4 + 29685\epsilon_5 + 2752\epsilon_6 - 251808\epsilon_7 - 331158\epsilon_8)\omega^{14} + (251808\epsilon_9 - 60364\epsilon_1 - 36522\epsilon_2 + 146634\epsilon_3 - 66261\epsilon_4 + 207565\epsilon_5 + 92376\epsilon_6 - 214784\epsilon_7 - 176002\epsilon_8)\omega^{12} + (214784\epsilon_9 - 451652\epsilon_1 - 146634\epsilon_2 + 145406\epsilon_3 - 326072\epsilon_4 + 114886\epsilon_5 + 62042\epsilon_6 + 878586\epsilon_7 + 143396\epsilon_8)\omega^{10} + (-878586\epsilon_9 - 376406\epsilon_1 - 145406\epsilon_2 + 49356\epsilon_3 - 310350\epsilon_4 - 34566\epsilon_5 - 10210\epsilon_6 + 741694\epsilon_7 - 723860\epsilon_8)\omega^8 +$

$(-741694\epsilon_9 - 76274\epsilon_1 - 49356\epsilon_2 + 5112\epsilon_3 - 99372\epsilon_4 - 28015\epsilon_5 - 12474\epsilon_6 + 207298\epsilon_7 - 708102\epsilon_8)\omega^6 + (-207298\epsilon_9 + 6674\epsilon_1 - 5112\epsilon_2 + 160\epsilon_3 - 9493\epsilon_4 - 3319\epsilon_5 - 1622\epsilon_6 + 20182\epsilon_7 - 217558\epsilon_8)\omega^4 + (-20182\epsilon_9 + 1526\epsilon_1 - 160\epsilon_2 - 276\epsilon_4 - 108\epsilon_5 - 56\epsilon_6 + 568\epsilon_7 - 21812\epsilon_8)\omega^2 - 568\epsilon_9 + 56\epsilon_1 - 624\epsilon_8\big),$

4. $L_4 = \frac{F}{G(\omega)}$, where
   - F is a polynomial of $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9$ and ω which has 192 different monomials.
   - $G(\omega)$ is an even polynomial of degree 40 which has 21 different monomials.
5. $L_5 = \frac{F}{G(\omega)}$, where
   - F is a polynomial of $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9$ and ω which has 336 different monomials.
   - $G(\omega)$ is an even polynomial of degree 70 which has 36 different monomials.

The origin critical point of system (2) is weak focus of order 4 if and only if

1. $\epsilon_7 = \frac{1}{\omega^2}(\epsilon_8 + \epsilon_9)$,
2. $\epsilon_1 = \frac{1}{8\,\omega^2 - 1}(3\omega^4\epsilon_3 + 4\omega^5\epsilon_5 + 2\omega^4\epsilon_6 - 4\omega^4\epsilon_8 - 3\omega^2\epsilon_2 - 6\omega^2\epsilon_4 - 2\omega^2\epsilon_5 - \omega^2\epsilon_6 + 6\omega^2\epsilon_8 - \epsilon_8)$,
3. $\epsilon_2 = \frac{1}{2(324\omega^6 - 315\omega^4 + 70\omega^2 + 1)}(648\omega^8\epsilon_3 + 144\omega^8\epsilon_6 - 630\omega^6\epsilon_3 - 720\omega^6\epsilon_4 - 324\omega^6\epsilon_5 - 488\omega^6\epsilon_6 - 648\omega^6\epsilon_8 + 140\omega^4\epsilon_3 + 604\omega^4\epsilon_4 + 315\omega^4\epsilon_5 + 364\omega^4\epsilon_6 + 630\omega^4\epsilon_8 + 2\omega^2\epsilon_3 - 119\omega^2\epsilon_4 - 70\omega^2\epsilon_5 - 60\omega^2\epsilon_6 - 140\omega^8\epsilon_8 - 15\epsilon_4 - \epsilon_5 - 2\epsilon_8)$
4. $\epsilon_4 = \frac{2(31104\omega^{14} + 3218\omega^{12} - 216174\omega^{10} + 250611\omega^8 - 116161\omega^6 + 20813\omega^4 - 959\omega^2 + 82)\omega^2\epsilon_6}{311040\omega^{14} - 984528\omega^{12} + 1188420\omega^{10} - 680538\omega^8 + 188699\omega^6 - 24554\omega^4 + 1526\omega^2 - 15}$

Since

$$\begin{vmatrix} \frac{\partial L_1}{\partial \epsilon_7} & \frac{\partial L_1}{\partial \epsilon_1} & \frac{\partial L_1}{\partial \epsilon_2} & \frac{\partial L_1}{\partial \epsilon_4} \\ \frac{\partial L_2}{\partial \epsilon_7} & \frac{\partial L_2}{\partial \epsilon_1} & \frac{\partial L_2}{\partial \epsilon_2} & \frac{\partial L_2}{\partial \epsilon_4} \\ \frac{\partial L_3}{\partial \epsilon_7} & \frac{\partial L_3}{\partial \epsilon_1} & \frac{\partial L_3}{\partial \epsilon_2} & \frac{\partial L_3}{\partial \epsilon_4} \\ \frac{\partial L_4}{\partial \epsilon_7} & \frac{\partial L_4}{\partial \epsilon_1} & \frac{\partial L_4}{\partial \epsilon_2} & \frac{\partial L_4}{\partial \epsilon_4} \end{vmatrix} = \frac{2\omega^6}{(\omega^2+1)^4(4\omega^2+1)^3(9\omega^2+1)^2(16\omega^2+1)}(311040\omega^{14} - 984528\omega^{12} + 1188420\omega^{10} - 680538\omega^8 + 188699\omega^6 - 24554\omega^4 + 1526\omega^2 - 15)$$

and is not equal to zero, then by perturbing the coefficients of Liapunov quantities, in the neighborhood of the critical point, four limit cycles can be bifurcated from the critical point at the origin of system (2).

Remark 2. By the same way, four limit cycles can be bifurcated from the origin of system (2) when the parameters satisfy condition τ_4 of Proposition 3.

Theorem 3. Three limit cycles can bifurcate from the critical point at the origin when the parameters in system (2) satisfy the conditions τ_2 or τ_3 of Proposition 3.

Proof. We suppose that parameters satisfy condition τ_2 of Proposition 3. When condition τ_2 holds, system (2) reduces to

$$\dot{x} = y$$
$$\dot{y} = z \quad (22)$$
$$\dot{z} = -\alpha\omega^2 x + \omega^2 y + \alpha z + a_5\omega^2 x^2 + \omega^2 a_6 xy + a_5 xz + a_6 yz,$$

and the transformed system (6) is obtained where

$$G(y_1, y_2, y_3) = -\frac{a_6}{\alpha} y_1 y_3 - \frac{a_5}{\alpha\omega} y_2 y_3 - \frac{(\alpha a_6 + a_5)}{\alpha^3 \omega} y_3^2 \quad (23)$$

It is easy to define the Liapunov function of F_o of system (6) which satisfies $\chi_o F_o = 0$:

$$\chi_o = (-\omega y_2 + G(y_1, y_2, y_3), \omega y_1 + \frac{\omega}{\alpha} G(y_1, y_2, y_3), \alpha y_3 - \alpha\omega G(y_1, y_2, y_3)),$$

where $G(y_1, y_2, y_3)$ is defined in (23) and

$$F_o = {y_1}^2 + {y_2}^2 + \sum_{k=3}^{N} \sum_{j=0}^{k} \sum_{i=0}^{j} C_{(k-j,j-i,i)} {y_1}^{k-j} {y_2}^{j-i} {y_3}^{i}.$$

We choose a point from center variety

$$(a_1, a_2, a_3, a_4, a_5, a_6, \alpha, \beta, \gamma) = (\omega^2, 0, 0, \omega^2, 1, 1, 1, -\omega^2, \omega^2)$$

We let
$a_1 = \omega^2 + \epsilon_1,\ a_2 = 0 + \epsilon_2,\ a_3 = 0 + \epsilon_3,$
$a_4 = \omega^2 + \epsilon_4,\ a_5 = 1 + \epsilon_5, a_6 = 1 + \epsilon_6,$
$\alpha = 1 + \epsilon_7,\ \beta = -\omega^2 + \epsilon_8,\ \gamma = \omega^2 + \epsilon_9,$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8$ and ϵ_9 are parameters after perturbation in the system. Transformation (5) is also used for perturbed part of vector field of system (6). Using MAPLE software and solving equation (18), the following linear independent terms of Liapunov quantities are obtained:

1. $L_1 = \frac{1}{\omega^2+1}(\omega^2\epsilon_7 - \epsilon_8 - \epsilon_9),$
2. $L_2 = \frac{1}{4(4\omega^4+5\omega^2+1)}(18\,\omega^6\epsilon_3 + 14\omega^4\epsilon_2 + 6\omega^4\epsilon_3 - 18\omega^4\epsilon_5 - \omega^4\epsilon_6 + 14\omega^4\epsilon_7 + 18\,\omega^2\epsilon_1 + 2\omega^2\epsilon_2 + \omega^2\epsilon_4 - 6\omega^2\epsilon_5 - \omega^2\epsilon_6 + 8\omega^2\epsilon_7 + 5\omega^2\epsilon_8 - 14\omega^2\epsilon_9 + 6\epsilon_1 + \epsilon_4 - \epsilon_8 - 8\epsilon_9),$
3. $L_3 = \frac{1}{8(36\omega^6+49\omega^4+14\omega^2+1)}(111\,\omega^8\epsilon_3 + 51\,\omega^6\epsilon_2 + 112\omega^6\epsilon_3 - 111\,\omega^6\epsilon_5 + 29\omega^6\epsilon_6 + 51\omega^6\epsilon_7 + 111\omega^4\epsilon_1 + 68\omega^4\epsilon_2 + 17\omega^4\epsilon_3 - 29\omega^4\epsilon_4 - 112\omega^4\epsilon_5 + 12\omega^4\epsilon_6 + 70\omega^4\epsilon_7 + 31\omega^4\epsilon_8 - 51\omega^4\epsilon_9 + 112\omega^2\epsilon_1 + \omega^2\epsilon_2 - 12\omega^2\epsilon_4 - 17\omega^2\epsilon_5 - \omega^2\epsilon_6 + 19\omega^2\epsilon_7 + 30\omega^2\epsilon_8 - 70\omega^2\epsilon_9 + 17\epsilon_1 + \epsilon_4 - \epsilon_8 - 19\epsilon_9),$
4. $L_4 = \frac{5}{64(576\omega^6+244\omega^4+29\omega^2+1)}(388\,\omega^8\epsilon_3 + 124\,\omega^6\epsilon_2 + 300\omega^6\epsilon_3 - 388\,\omega^6\epsilon_5 + 175\omega^6\epsilon_6 + 124\omega^6\epsilon_7 + 388\omega^4\epsilon_1 + 244\omega^4\epsilon_2 + 32\omega^4\epsilon_3 - 175\omega^4\epsilon_4 - 300\omega^4\epsilon_5 + 54\omega^4\epsilon_6 + 158\omega^4\epsilon_7 + 89\omega^4\epsilon_8 - 124\omega^4\epsilon_9 + 300\omega^2\epsilon_1 - 54\omega^2\epsilon_4 - 32\omega^2\epsilon_5 - \omega^2\epsilon_6 + 34\omega^2\epsilon_7 + 88\omega^2\epsilon_8 - 158\omega^2\epsilon_9 + 32\epsilon_1 + \epsilon_4 - \epsilon_8 - 34\epsilon_9)).$

The origin critical point of system (2) is weak focus of order 3 if and only if

1. $\epsilon_7 = \frac{1}{\omega^2}(\epsilon_8 + \epsilon_9),$

2. $\epsilon_1 = \frac{-1}{6(3\omega^2+1)}$ $(18\omega^6\epsilon_3 + 14\omega^4\epsilon_2 + 6\omega^4\epsilon_3 - 18\omega^4\epsilon_5 - \omega^4\epsilon_6 + 2\omega^2\epsilon_2 + \omega^2\epsilon_4 - 6\omega^2\epsilon_5 - \omega^2\epsilon_6 + 19\omega^2\epsilon_8 + \epsilon_4 + 7\epsilon_8)$,
3. $\epsilon_2 = \frac{1}{4\omega^2(159\omega^6+65\omega^4+9\omega^2+7)}$ $(633\omega^8\epsilon_6 - 633\omega^6\epsilon_4 + 613\omega^6\epsilon_6 - 633\omega^6\epsilon_8 - 613\omega^4\epsilon_4 + 183\omega^4\epsilon_6 - 613\omega^4\epsilon_8 - 183\omega^2\epsilon_4 + 11\omega^2\epsilon_6 - 183\omega^2\epsilon_8 - 11\epsilon_4 - 11\epsilon_8)$.

Since

$$\begin{vmatrix} \frac{\partial L_1}{\partial \epsilon_7} & \frac{\partial L_1}{\partial \epsilon_1} & \frac{\partial L_1}{\partial \epsilon_2} \\ \frac{\partial L_2}{\partial \epsilon_7} & \frac{\partial L_2}{\partial \epsilon_1} & \frac{\partial L_2}{\partial \epsilon_2} \\ \frac{\partial L_1}{\partial \epsilon_7} & \frac{\partial L_1}{\partial \epsilon_1} & \frac{\partial L_1}{\partial \epsilon_2} \end{vmatrix} = \frac{-\omega^4(159\omega^5 + 65\omega^4 + 9\omega^2 + 7)}{8(\omega^2+1)(4\omega^4+5\omega^2+1)(36\omega^6+49\omega^4+14\omega^2+1)}$$

and it is not equal to zero, then by perturbing the coefficients of Liapunov quantities, in the neighborhood of the critical point, three limit cycles can be bifurcated from the critical point at the origin of system (2).

Remark 3. By the same way, three limit cycles can be bifurcated from the origin of system (2) when the parameters satisfy condition τ_3 of Proposition 3.

4. CONCLUSIONS

The centre bifurcation of a third order differential equation (1) is studied by using a simple technique to estimate the cyclisity bifurcating from centre (see (Salih, 2015) and (Salih & Hasso, 2017)). Four sets of sufficient condition of parameters for the existence of a centre are obtained. When we perturbed the parameters, by taking the linear parts of the corresponding Liapunov quantities of the perturbed system, a number of bifurcated periodic orbits have appeared. As a result, four limit cycles can be bifurcated from two sets of condition and three limit cycles from the other two sets of condition.

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