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RESEARCH PAPER

Periodic Solutions Bifurcating From a Curve of Singularity of the Jerk System

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A B S T R A C T:

We investigate a periodic solution which bifurcates from a curve of the singularity of the jerk system inR^3.More precisely, we give the explicit states for the existence of a periodic solution of the jerk system with a nonisolated singular point, where for each singular point has a simple pair of purely imaginary and one zero eigenvalues. We recall for this point of singularity as a zero-Hopf (z-H) singular point. The coefficients in the jerk system are described for which the z-H singularity occur at each point of that curve of singularity. We show that for each point at that curve of singularity there is only one family of parameters which exhibits such type of singular points. The method of averaging in the second order is utilized to determine one periodic solution which bifurcates from any point of that curve of singularity. As far as, we realize that this investigation is the study on bifurcations from a curve of nonisolated z-H singularity to provide a periodic solution via the method of averaging. Under a generic small perturbation at the parameters, we prove that a periodic solution will be bifurcated at any point that located on a curve of a singularity of the jerk system.

KEY WORDS: Jerk system, periodic orbit, zero-Hopf singularity, the method of averaging
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INTRODUCTION

Time derivative of acceleration in physics is called the jerk. It can be defined in dynamical system by $\dddot{x} = f(x, \dot{x}, \ddot{x})$, see (Gottlieb, 1998). This equation, by changing the variables, can be remold into a general 3D differential system

$$\dot{x} = y, \ \dot{y} = z, \ \dot{z} = g(x, y, z). \tag{1}$$

System (1) is very remarkable in nonlinear dynamic systems. For instance, the simplest possible chaotic system is in this form studying in (Sprott, 1997).

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$$g(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 xz + a_9 yz,$$

where a_i for $i = 0, \ldots, 9$ are real parameters. This system has only one singular point when

$$a_0 = \frac{a_1^2}{4a_4}, \ a_4 \neq 0 \quad \text{or} \quad a_4 = 0.$$

The case $a_0 = \frac{a_1^2}{4a_4}, a_4 \neq 0$ was considered by Wei et al. in (Wei, Sprott, &Chen, 2015). They addressed a periodic solution which bifurcates from one single non-hyperbolic singular point. The other case $a_4 = 0$ was considered by Molaie, et al. in (Molaie, Jafari, Sprott, & Golpayegani, 2013). They found 23 simple chaotic flows of system (1). Here, the case $a_4 = 0$ with extra conditions will be considered (see Propositions 1 and 2) to determine nonisolated zero–Hopf singularity which fill the curve passing through the origin of coordinates.

A zero-Hopf singular point of a polynomial differential system in the 3-dimentions is an isolated singular point which owns a zero and a simple pair of purely imaginary eigenvalues. This type of singular point is an interesting topic and it considered by (John, 1981; John & Philip, 2013; Kuznetsov, 2013; Rizgar, 2017) and others. Generally, a zero– Hopf bifurcation is a 2-parameter unfolding of a polynomial systems in the 3-dimensions which has an isolated z-H singular point. It has realized that several intricate sets of invariant of the unfolding maybe bifurcate in a neighborhood of an isolated z-H singular point under some generic states, for example a local birth of "chaos" could be implied from the z-H bifurcation, see (Baldom´a & Seara, 2006; Broer & Vegter, 1984).

In general, there is no theory to determine when some periodic solutions are bifurcated by perturbing the parameters of the 3D systems from the z-H singular point. Some authors are investigated on zero–Hopf singular point, see (Garc´ıa, Llibre, & Maza, 2014; Llibre, 2014; Llibre, Makhlouf, & Badi, 2009; Euz´ebio, Llibre, & Vidal, 2015; Euz´ebio & Llibre, 2017; Llibre &P´erez-Chavela, 2014; Castellanos, Llibre, & Quilantan, 2013; Llibre, Oliveira, & Valls, 2015; Rizgar, 2017). They studied the periodic solutions bifurcating in a neighborhood of the isolated z-H singular point. Only the two works (Llibre & Xiao, 2014; Cˆandido & Llibre, 2018), they studied the periodic solutions bifurcating in a neighborhood of a nonisolated zero–Hopfpoint located only at (0,0,0)-point. However, a case where system (1) has infinitely many (actually a continuum of) zero-Hopf singularity will be studded. This is explained in Proposition 1. By linear algebra the following result can be obtained.

System (1) has a curve of a singularity which passes through (0,0,0)-point if and only if

$$a_0 = a_1 = a_4 = 0.$$

From now, we consider the condition $a_0 = a_1 = a_4 = 0$. So, we modify the general case in system (1) into following one

$$\dot{x} = y,\ \dot{y} = z, \dot{z} = f(x,y,z), \qquad (2)$$

$$f(x,y,z) = a_2y + a_3z + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz,$$

where there are no x and x^2 terms in the $\dot{z}$ equation to guarantee that there is a curve of singularity. We see that any singular point (p_0, p_1, p_2) of system (2) must have $p_1 = p_2 = 0$, and eigenvalues λ that satisfy

$$\lambda^3 - f_z\lambda^2 - f_y\lambda - f_x$$

in which $f_x = 0, f_y = a_2 + a_7p_0$ and $f_z = a_3 + a_8p_0$. Using the Routh–Hurwitz stability criterion, for the singular point (p_0, p_1, p_2) to be asymptotically stable, we need

$$f_z < 0,\ f_yf_z + f_x > 0, \text{and } f\_x < 0.$$

Therefore, the quadratic jerk system (2) cannot have the stable equilibria. As shown in (Llibre & Xiao, 2014), we can easily find that system (2) has infinitely many (actually a continuum of) singular points which are at the following curve (curve of a singularity)

$$C_x = \{(p_0, 0,0) : p_0 \in \mathbb{R}\}. \qquad (3)$$

An analysis is made on the polynomial system (1) such that each point of the curve C_x becomes a nonisolated z-H singular point which is the main purpose in this work. This is described in following proposition. It has shown that there is only one family of parameters in the jerk system in which any singular point at the curve$C_x$becomes the z-H singular point. Furthermore, we use the method of averaging in the second order to estimate a periodic solution of the jerk system which bifurcates at the line of singularity.

Proposition 1. System (2) has the z-H singular points which are localized at the curveC_x, if the following conditions are satisfied

$$a_3 = -a_8p_0, \text{and } a_2 + a_7p_0 < 0. \qquad (4)$$

We prove this proposition in Section 2.

In Section 1, we describe the method of averaging in the second order. It explains the accurate conditions for the existence of a periodic solution which bifurcates from the nonisolated z-H singular point. In Section 2, the bifurcations with nonisolated z-H singular points are studied using a small perturbation of system (2) keeping the nonisolated singular points at the curve C_x which obtains a perturbation system. In the neighborhood of any point, the given perturbation system is reduced to a 2π-periodic system in a type of cylindrical coordinates, and the re-scaling of variables is needed to prove the main result (Theorem 1).

The following theorem is our main result to obtain a periodic solution which bifurcates from one family of parameters for the jerk system with a curve of z-H singularity.

Theorem 1 Let $(a_2, a_3, a_5, a_6, a_7, a_8, a_9) = (b_2 + \epsilon\, c_2, \epsilon c_3, b_5 + \epsilon\, c_5, b_6 + \epsilon\, c_6, \epsilon\, c_7, \epsilon\, c_8, b_9 + \epsilon\, c_9)$ be a vector and $c_3 = 0, b_5 = -\omega^2 b_6$, with a sufficiently small parameterϵ. If $b_6 b_7 \neq 0$. Then, system (2) has a z-H bifurcation at the nonisolated singular point which localizes at the curve of singularity, and a periodic solution can be produced for each point at that curve of a singularity when $\epsilon = 0$.

We prove Theorem 1 in Section 2. The method of averaging of second order is the main tool to prove Theorem 1. The method of averaging is made a history and for a new explanation of this method, one can see the work of (Sanders et al, 2007). In the following section (Section 1), we recall the method of averaging of second order as was described in (Buica, Fran¸coise, & Llibre, 2007; Pi & Zhang, 2009).

1. THE METHOD OF AVERAGING OF THE SECOND ORDER FOR PEIRODIC SOLUTIONS

The purpose of this section is to describe basic results from the method of averaging. This method requires to prove the bifurcating periodic solutions from nonisolated z-H singular points of system (2), for a proof of the method of averaging on can found in Theorem 2.6.1 of Sanders and Verhulst (O'Malley Jr, 1987) also Theorem 11.5 of Verhulst. The averaging method of second order was described clearly in (Buica et al., 2007; Llibre et al., 2009). Also, in (Marsden & McCracken, 2012; Chow & Hale, 2012; Buica et al., 2007; Sanders, Verhulst, & Murdock, 1987) the researchers have devoted their attempt to determine a periodic solution using the method of averaging.

The following theorem is summarized the method of averaging of second order.

Theorem 2 Consider the Differential equation

$$\dot{x} = \epsilon\, f_1(t, x) + \epsilon^2 f_2(t, x, \epsilon) + \epsilon\, R(t, x, \epsilon) \qquad (6)$$

where $f_1, f_2: \mathbb{R} \times Y \to \mathbb{R}^n$, $R: \mathbb{R} \times U \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^n$ are continuous functions, T-periodic in the variable t and $U \subset \mathbb{R}^n$ is an open subset. Assume that the following conditions satisfy

i. $f_1(t, .) \in C^1(U), \forall t \in \mathbb{R}$, f_2, R and $\frac{\partial f_1}{\partial x}$ are locally Lipschitz with respect to variable x, and R is twice differentiable with respect to x .

ii. Define $F_{i0} : U \to \mathbb{R}^n$ for $i = 1,2$ by

$$F_{10} = \frac{1}{T}\int_0^T f_1(s, z)\, ds$$

$$F_{20} = \frac{1}{T}\int_0^T [\, D_z f_1(s, z) \int_0^s f_1(s, z)\, ds + f_2(s, z)]\, ds$$

where $D_z f_1\,(s, z)$ is the Jacobian determinant matrix of the components of $f_1$with respect to z.

iii. For V bounded and an open set in U, for $\epsilon \in (-\epsilon_0, \epsilon_0)\backslash\{0\}$ there is $r_\epsilon \in V$ such that$F_{10} + \epsilon\, F_{20} = 0$ and$d_B(F_{10} + \epsilon\, F_{20})$ is not equal to zero.

Hence, for a sufficiently small$|\epsilon| > 0$, a T-periodic solution $\phi(\Delta, \epsilon)$ of system (6) is existed such that $\phi(0, \epsilon) = r_\epsilon$.

The term$d_B(F_{10} + \epsilon\, F_{20}) \neq 0$ denotes the Brouwer degree of the function$F_{10} + \epsilon\, F_{20}: V \to \mathbb{R}^n$ at the singular point r which is not finish. A sufficient condition for the inequality to be true is that the Jacobian of$(F_{10} + \epsilon\, F_{20})$ at r_ϵ is notfinish. If$F_{10} \neq 0$, then the zeros of$(F_{10} + \epsilon\, F_{20})$ are mainly the zeros of $F_{10}$for $0 \ll \epsilon$. In this situation, the previous result uses the method of averaging of first order. If$F_{10} = 0$and$F_{20} \neq 0$, then the zeros of$F_{10} + \epsilon\, F_{20}$are mainly the zeros of F_{20} for$0 \ll \epsilon$. In this situation, the previous result gives the method of averaging of second order.

2. PROOF OF THE MAIN RESULT

Proof of Proposition 1. The characteristic equation of system (2) for each singular point, say $(p_0, 0,0)$, which is at the curve L is

$$\lambda^3 - (a_3 + a_8 p_0)\lambda^2 - (a_2 + a_7 p_0)\lambda = 0.$$

Putting this in $\lambda(\lambda^2 + \omega^2) = 0$where $\omega > 0$, we obtain only one family of condition, that is $a_3 = -a_8 p_0$ and $a_2 + a_7 p_0 < 0$.

Therefore, for any nonisolated singular point $(p_0, 0,0) \in L$ of system (2) has the eigenvalues$\lambda_1 = 0$ and $\lambda_{2,3} = \pm\sqrt{a_2 + a_7 p_0}$. This means that each singular point is in the z-H singular type.

Proof of Theorem 1. If we perturb the parameters $(a_2, a_3, a_5, a_6, a_7, a_8, a_9) = (b_2 + \epsilon\, c_2, \epsilon c_3, b_5 + \epsilon\, c_5, b_6 + \epsilon\, c_6, \epsilon\, c_7, \epsilon\, c_8, b_9 + \epsilon\, c_9)$ with $0 \ll \epsilon < 1$.Then system (2) becomes

$$\dot{x} = y,$$
$$\dot{y} = z, \qquad (7)$$
$$\dot{z} = (b_2 + \epsilon\, c_2)y + \epsilon c_3 z + (b_5 + \epsilon\, c_5)y^2 + (b_6 + \epsilon\, c_6)z^2 + (b_7 + \epsilon\, c_7)xy + \epsilon\, c_8 xz + (b_9 + \epsilon\, c_9)\, yz.$$

Re-scaling variables $(x, y, z) = (\epsilon X, \epsilon Y, \epsilon Z)$ system (7) in variables (X, Y, Z) becomes

$$\dot{X} = Y,$$
$$\dot{Y} = Z, \qquad (8)$$
$$\dot{Z} = (b_2 + \epsilon\, c_2)Y + \epsilon c_3 Z + (\epsilon\, b_7 + \epsilon^2 c_7)XY + \epsilon^2 c_8 XZ + (c_5\epsilon^2 + b_5\epsilon)Y^2 + (c_9\epsilon^2 + b_9\epsilon)ZY + (\epsilon b_6 + \epsilon^2 c_6)Z^2.$$

The above system has also a line of singular points (X = X, Y = 0, Z = 0). Thus, the linear part at each of nonisolated singular point of system (8) when $\epsilon = 0$ shall be transformed to its real Jordan form, which is as

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\omega^2 = -b_2$. For doing that changing of variables

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -\omega & 0 \\ -\omega^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}, \qquad (9)$$

can be used. In the new variables (U, V, W), system (8) writes

$$\dot{U} = -\omega V + \Big(V^2 b_5 - c_3\omega U + b_9\omega\, UV + \omega^2 b_6 U^2 - \tfrac{1}{\omega} V\,(c_2 + b_7 U + b_7 W)\Big)\epsilon + \Big(-\,\omega^2 U^2 c_6 - c_5 V^2 - \omega\, c_9 VU + c_8 U(U + W) + \tfrac{1}{\omega} c_7 V(U + W)\Big)\epsilon^2,$$
$$\dot{V} = \omega U, \qquad (10)$$
$$\dot{W} = \Big(c_5 V^2 - c_8 U(U + W) + \omega U(c_6\omega U + c_9 V) - \tfrac{1}{\omega} c_7 V(U + W)\Big)\epsilon^2 + \Big(-c_3 U + b_5 V^2 + \omega(b_9 UV + \omega b_6 U^2) - \tfrac{1}{\omega}(b_7 VW + b_7 UV + c_2 V)\Big)\epsilon.$$

Using the cylindrical coordinates (r, θ, W)defining by

$$(U, V, W) = (\, r cos(\theta), r sin(\theta), W),$$

and suggesting θ as the independent variable. System (10) can be expressed as

$$\frac{dr}{d\theta} = \epsilon\, f_{1,1} + \epsilon^2 f_{1,2} + O(\epsilon),$$
$$\frac{dW}{d\theta} = \epsilon\, f_{2,1} + \epsilon^2 f_{2,2} + O(\epsilon), \qquad (11)$$

where

$$f_{1,1} = \frac{r\,cos(\theta)}{\omega^2}(r\omega(\omega^2 b_6 - b_5)\,cos^2(\theta) + (r\,sin(\theta)(b_9\omega^2 - b_7) - \omega c_3)cos(\theta) + (-b_7 W - c_2)sin(\theta) + r\omega b_5),$$

$$f_{1,2} = \frac{rcos(\theta)}{\omega^2}\,(-\,(r\,cos\,(\theta)(\,\omega^2\, r\,cos\,(\theta)sin(\theta)c_9 + r\omega c_5 - \big(cos(\theta)\big)^2\, r\omega c_5 + \omega^3 r\,\big(cos(\theta)\big)^2\, c_6 - c_8\,\omega cos(\theta)W - c_8\omega r\big(cos(\theta)\big)^2 - rsin(\theta)c_7\, cos\,\theta - sin(\theta)c_7 W) + r cos\,\frac{\theta}{\omega^2}\Big(\,r\,\omega b_5 - \big(cos(\theta)\big)^2\, r\,\omega b_5 + r\,\omega^2 cos(\theta)sin(\theta)b_9 - sin(\theta)b_7 W - sin(\theta)c_2 - c_3\,\omega\,cos(\theta) - rsin(\theta)b_7 cos(\theta) + r\omega^3\big(cos(\theta)\big)^2 b_6\Big)\Big(\,rb_7\,cos(\theta) +$$

$sin(\theta)\left(cos(\theta)\right)^2 r\omega b_5 - r\,\omega\, b_5\, sin\,(\theta) - r\,\omega^2 cos(\theta)b_9 - sin(\theta)r\omega^3\left(cos(\theta)\right)^2 b_6 + sin(\theta)c_3\omega\, cos(\theta) + b_7W - \left(cos(\theta)\right)^2 b_7W + c_2 + \left(cos(\theta)\right)^3 r\,\omega^2\, b_9 - \left(cos\,(\theta)\right)^3 rb_7 - \left(cos(\theta)\right)^2 c_2\Big)\,\omega^2),$

$f_{2,1} = \frac{r}{\omega^3}(r\,\omega\,(\omega^2 b_6 - b_5)(\,cos(\theta))^2 + (r(-b_7 + \omega^2 b_9)sin(\theta) - c_3\,\omega)\,cos(\theta) + (\,-b_7W - c_2)sin(\theta) + r\,\omega b_5),$

$f_{22} = \Big(r\,\Big(\,\omega^2 rcos(\theta)sin(\theta)c_9 + r\omega c_5 - \left(cos(\theta)\right)^2 r\omega c_5 + \omega^3 r\left(cos(\theta)\right)^2 c_6c_8\omega cos(\theta)W - c_8\omega r\left(cos(\theta)\right)^2 - rsin(\theta)c_7cos(\theta) - sin(\theta)c_7W\Big) - \frac{r}{\omega}\Big(\,r\omega b_5 - \left(cos(\theta)\right)^2 r\omega b_5 + r\omega^2 cos(\theta)sin(\theta)b_9 - sin(\theta)b_7W - sin(\theta)c_2 - c_3\omega cos(\theta) - rsin(\theta)b_7cos(\theta) + r\omega^3\left(cos(\theta)\right)^2 b_6\Big)\Big(\,-rb_7cos(\theta) - sin(\theta)\left(cos(\theta)\right)^2 r\omega b_5 + r\omega b_5 sin(\theta) + r\omega^2 cos(\theta)b_9 + sin(\theta)r\omega^3\left(cos(\theta)\right)^2 b_6 sin(\theta)c_3\omega cos(\theta) - b_7W + \left(cos(\theta)\right)^2 b_7W - c_2 - \left(cos(\theta)\right)^3 r\omega^2 b_9\left(cos(\theta)\right)^3 rb_7 + \left(cos(\theta)\right)^2 c_2\,\Big)\,\omega^2\Big) - \frac{r}{\omega^2}\Big(\,r\omega b_5 - \left(cos(\theta)\right)^2 r\omega b_5 + r\omega^2\, cos(\theta)sin(\theta)b_9 - sin(\theta)b_7W - sin(\theta)c_2 - c_3\omega cos(\theta) - rsin(\theta)b_7cos(\theta) + r\omega^3\left(cos(\theta)\right)^2 b_6\Big)\Big(-rb_7cos(\theta) - sin(\theta)\left(cos(\theta)\right)^2 r\omega b_5 + r\omega b_5 sin(\theta) + r\omega^2 cos(\theta)b_9 + sin(\theta)r\omega^3\left(cos(\theta)\right)^2 b_6 - sin(\theta)c_3\omega cos(\theta) - b_7W\left(cos(\theta)\right)^2 b_7W - c_2 - \left(cos(\theta)\right)^3 r\omega^2 b_9 + \left(cos(\theta)\right)^3 rb_7 + \left(cos(\theta)\right)^2 c_2\Big)\,\omega^3\,.$

The two systems (11) and (6) are equivalent by taking the notation in Theorem 2, that is, letting $t = \theta,\ T = 2\pi,\ x = (r, W) \in (0,\infty) \times R,\ x_0 = (ro, Wo)$ and

$$F_{10}(t,x) = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta, r, W)d\theta = \begin{bmatrix} \frac{1}{2\pi}\int_0^{2\pi} f_{1,1}(\theta, r, W)d\theta \\ \frac{1}{2\pi}\int_0^{2\pi} f_{2,1}(\theta, r, W)d\theta \end{bmatrix} = \begin{bmatrix} f^{\circ}_{1,1}(r, W) \\ f^{\circ}_{2,1}(r, W) \end{bmatrix}. \quad (12)$$

Computing equation (12), we obtain

$$f^{\circ}_{1,1}(r, W) = -\frac{1}{2}\frac{r\,c_3}{\omega},\ \ f^{\circ}_{2,1}(r, W) = \frac{r^2(\omega^2 b_6 + b_5)}{2\omega}. \quad (13)$$

From (Buica et al., 2007) the non-zero solution of system (13) gives a periodic solution which bifurcates in the neighborhood of the each point at the curve $L$for system (2). In other word, for $r > 0$, the method of averaging of first order characterized by (Llibre & P´erez Chavela, 2014) does not give the possible periodic solutions bifurcating from the z-H singular point. Therefore, the averaged function of the first order $(f^{\circ}_{1,1}(r, W), f^{\circ}_{2,1}(r, W))$ is equal to zero if and only if

$c_3 = 0$, and $b_5 = -\omega^2 b_6$ $(\omega^2 = -b_2)$. (14)

For using the Brouwer degree of second order, we now consider the conditions in equation (14) to use the method of averaging of second order. Therefore, the following expression must be computed:

$$\begin{bmatrix} \frac{\partial f_{1,1}}{\partial r} & \frac{\partial f_{1,1}}{\partial W} \\ \frac{\partial f_{2,1}}{\partial r} & \frac{\partial f_{2,1}}{\partial W} \end{bmatrix}\begin{bmatrix} \int_0^{\theta} f_{1,1}(\theta, r, W)d\theta \\ \int_0^{\theta} f_{2,1}(\theta, r, W)d\theta \end{bmatrix} + \begin{bmatrix} f_{2,1} \\ f_{22} \end{bmatrix}. \quad (15)$$

We integrate the above expression from 0 to 2π with respect θ and divide by 2π to obtain the following equation

$$F_{20}(t,x) = \begin{bmatrix} F^{*}_{2,0,1} \\ F^{*}_{2,0,2} \end{bmatrix} + \begin{bmatrix} \frac{r}{8\omega}(r^2 b_7 b_6 + 4c_8W) \\ \frac{-r^2}{2\omega}(c_8 - c_5 - \omega^2 c_6 + c_2 b_6 + b_6 b_7 W) \end{bmatrix}. \quad (16)$$

To look for a periodic solution, we solve the following system of equation (We keep away for the zero solution with $r \geq 0$),

$$F_{20}(t,x) = 0 \rightarrow \begin{bmatrix} F^{\circ}_{2,0,1} = 0 \\ F^{\circ}_{2,0,2} = 0 \end{bmatrix}. \quad (17)$$

That is

$$(r, W) = \left(\frac{2\sqrt{c_8(c_2b_6 - c_5 + c_8 - c_6\omega^2)}}{b_7b_6}, \frac{c_5 + c_6\omega^2 - c_2b_6 - c_8}{b_7b_6}\right)$$

Let (r^*, W^*) be a solution of system (17). In order to have a periodic solution according with Theorem 2, we must have

$$J(F_{2,0}(r, W))|_{(r,w)=(r^*,W^*)} \neq 0. \quad (18)$$

We then compute the Jacobian matrix of (17) to obtain

$$-\frac{2c_8^2}{b_7^2b_6^2}(\omega^2c_6 - b_6c_2)^2,$$

which is not equal to zero when $b_7b_6 \neq 0$, where c_8, c_2 and c_6 are arbitrary, then we can chose that $c_8(\omega^2c_6 - b_6c_2)^2 \neq 0$.

Shortly, the solution (r^*, W^*) of system (17) which confirms condition (18) satisfies the assumptions (i) and (ii) of Theorem 2. So, for utilizing the method of averaging of second order we deduce that system (11) has the periodic solution bifurcating at each point at the curve L. Therefore, due to the re-scaling in system (8) with generic conditions (5) and (14), the periodic solution of system (2) is obtained which bifurcates from the nonisolated z-H singularity.

Therefore, for $\epsilon > 0$ sufficiently small, Theorem 2 guaranties that there is a periodic solution $(r(\theta,\epsilon), W(\theta,\epsilon))$ of system (11) such that$\big(r(0,\epsilon), W(0,\epsilon)\big) \to (r^*, W^*)$ when $\epsilon \to 0$. Thus, for $\epsilon \gg 0$

$$(U(\theta,\epsilon), V(\theta,\epsilon), W(\theta,\epsilon)) = (r\cos(\theta), r\sin(\theta), W(\theta,\epsilon)), \quad (19)$$

is the periodic solution of system (10). Consequently, system (8) under the change of variables in equation (9) has the periodic solution $(X(\theta), Y(\theta), Z(\theta))$ which obtains from (19). Finally, for $\epsilon > 0$sufficiently small, we have $(x(\theta), y(\theta), z(\theta)) = (\epsilon X(\theta), \epsilon Y(\theta), \epsilon Z(\theta))$,

Thus, system (2) has the periodic solution tending to the point at the curve L when $\epsilon \to 0$. It is the periodic solution which starts at the z-H singular point localized at the curve L when $\epsilon \to 0$. Hence, we complete the proof this theorem.

3. CONCLUSIONS

In this work, the jerk system has been considered which was suggested by Gottlieb in (Gottlieb, H. P. 1998). We put conditions on parameters in which the jerk system has a curve of singularity. We have then described the values of the parameters for which a zero-Hopf singular point occurs at the point of that curve of singularity. Moreover, we have used the averaging method of second order (see Theorem 2) to estimate of a periodic solution which bifurcates from the zero–Hopf equilibrium point. This theorem obtained a periodic solution $\big(r(\theta,\epsilon), W(\theta,\epsilon)\big)$ of the jerk system such that

$$\big(r(0,\epsilon), W(0,\epsilon)\big) = (r^*, W^*) + O(\epsilon),$$

where

$$(r^*, W^*) = \left(\frac{2\sqrt{c_8(c_2b_6 - c_5 + c_8 - c_6\omega^2)}}{b_7b_6}, \frac{c_5 + c_6\omega^2 - c_2b_6 - c_8}{b_7b_6}\right).$$

In coordinates (r, θ, ω), we can write this periodic solution as

$$\big(r(t,\epsilon), \theta(t,\epsilon), \omega(t,\epsilon)\big)$$

which satisfies

$$\big(r(0,\epsilon), \theta(0,\epsilon), W(0,\epsilon)\big) = (r^*, 0, W^*) + O(\epsilon).$$

Now, in the coordinates (U, V, W) the periodic solution becomes

$$(U(t,\epsilon), V(t,\epsilon), W(t,\epsilon))$$

which satisfies

$$(U(0,\epsilon), V(0,\epsilon), W(0,\epsilon)) = (r^*, 0, W^*) + O(\epsilon).$$

Also, the above periodic solution in the variables (X, Y, Z) becomes $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$ which satisfies

$$(X(0,\epsilon), Y(0,\epsilon), Z(0,\epsilon)) = (r^* + W^*, 0, -\omega^2r^*) + O(\epsilon).$$

Finally, in the variables (x, y, z) this periodic solution becomes

$$(x(t,\epsilon), y(t,\epsilon), z(t,\epsilon))$$

which satisfies

$$\big(x(0,\epsilon), y(0,\epsilon), z(0,\epsilon)\big) = \epsilon(r^* + W^*, 0, -\omega^2r^*) + O(\epsilon).$$

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