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RESEARCH PAPER

# Free vibration analysis of multi-cracked nanobeam using nonlocal elasticity theory

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**A B S T R A C T:**

The aim of this paper is to study the free lateral vibration of multi-cracked nanobeams, and consequently finding the natural frequencies of the cracked nanobeams using two methods. The model of the beam is Euler-Bernoulli in which shear effect has been neglected. Crack is assumed to divide the beam into two segments and these segments are connected to each other by a linear spring and a rotational spring. The crack induces more flexibility to the beam and reduces the stiffness of the beam and consequently influences the dynamic response and the natural frequencies of the beam. Cases of double-cracked and triple-cracked nanobeams are studied. It is observed that when the number of the cracks are increased, the natural frequencies will be decreased. Nonlocal elasticity theory is exposed to the equation of motion. Nonlocal parameter and number of the cracks affect the natural frequencies of the nanobeams. For the case of cantilever, the results are slightly different in contrast to simply supported and clamped-clamped cases. It has been shown that some frequency modes remain constant when the crack severity increases, because of the location of the crack which is a node for a certain mode of vibration.

Keywords: Free vibration, Multi-cracked nanobeams, Euler-Bernoulli, Nonlocal elasticity theory.
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## 1. INTRODUCTION

Nano-sized structures are being applied to highly sensitive and very fine devices, sensors and electromechanical systems. These nanostructures could be plates, beams, or other membranes [1, 2]. When structure is in nano-dimension, the word of size effect will be highlighted and it could not be neglected during analysis of the micro- or nano-sized structures.

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There are several theories of continuum mechanics that have paid attention to the size effect. These theories are size dependent such as modified couple stress theory (MCST), couple stress theory(CST), strain gradient theory, and nonlocal elasticity theory that could be used for analysis of micro and nanostructures [3-6]. Among these continuum theories, nonlocal elasticity theory is one of the widely used theories [7]. In nonlocal theory the size effect is an important factor and it enters the equations for analyzing the wave propagation, crack, and dislocation problems [8, 9].

Nonlocal continuum theory has simpler calculations in contrast to molecular dynamics and discrete atomic simulations. For the first time, Peddieson et al. [10] used nonlocal continuum theory in nanotechnology and derived the equations of the nonlocal Euler- Bernoulli beam for the case of static. Later, Zhang et al.[11] extended the derived equations to the dynamic problems. Lu et al. [12] has proposed the general expression of the shear force and bending moment for Euler-Bernoulli beam using nonlocal elasticity. Wang [13], Wang et al. [14], and Wang and Varadan [15], have obtained equations for Timoshenko beam using nonlocal elasticity based on the nonlocal bending moment and the local shear force, in which the distributed transverse force was not considered. Reddy [16] derived equations of motion for all kinds of the well-known beam theories such as Euler-Bernoulli, Timoshenko, and Reddy, in order to obtain analytical and numerical solutions on static deflections, buckling loads, and natural frequencies by using the nonlocal elasticity theory relations.

There are several researchers that have used the different continuum theories for linear and nonlinear vibration analysis of nanorods and nanobeams including several parameter effects [17-21].

Loya et al. [22] proposed two methods to analyze free vibration of the nanobeams. They obtained the natural frequencies of the single cracked Euler nanobeams using nonlocal elasticity. Their two proposed methods give the same results the first one has longer calculations but second one leads to shorter equations to find natural frequencies.

Torabi and Nafar Dastgerdi [23] studied the free vibration of cracked Timoshenko nanobeams to find the natural frequencies of the single-cracked nanobeams using nonlocal elasticity. Their results of Timoshenko well agreed with the Euler-Bernoulli beam results.

Roostai and Haghpanahi [24] studied the free vibration of multi-cracked nanobeams by a different method in which the induced flexibility due to the crack, was used instead of crack severity in calculations. Loghmani and Yazdi [25] studied free lateral vibration of Euler-Bernoulli nanobeam with multiple discontinuities. Cracks and steps were considered as discontinuities. Based on wave approach, vibrations were assumed as moving waves along the structure. Mahdi Soltanpour and co-worker's [26] studied free transverse vibration analysis of size dependent Timoshenko FG cracked nanobeams resting on elastic medium. Ebrahimi and Mahmoodi [27] studied the thermal loading effect on free vibration characteristics of carbon nanotubes (CNTs) with multiple cracks. Furthermore, a noticeable amount of studies has been conducted on the case of forced vibration analysis. Akbaş [28] worked on the forced vibration responses of functionally graded Timoshenko nanobeam using modified couple stress theory with damping effect.

## 2. THEORY AND FORMULATION

### 2.1. Governing equations for the Eringen nonlocal elasticity theory

According to the nonlocal elasticity theory [7], the nonlocal stress-tensor ($\sigma_{ij}$) at point x in a body is not only a function of the strain at the same point (local theory), but it is also a function of strains at all other points of the structure. As for the case of homogenous and isotropic nonlocal elastic solid, the general form of equations is written as

$$\sigma_{ij}(x) = \int \alpha(|\acute{x} - x|, T) t_{ij}(x) dV(\acute{x}) \qquad (1)$$

The kernel $\alpha|\acute{x} - x|$ is the nonlocal modulus which incorporates into the constitutive relation the nonlocal effect of the stress at point x created by local strain at the point $\acute{x}$ . $|\acute{x} - x|$is the Euclidean distance. The expressions $t_{ij}$ are the components of the classical local stress tensor at point x. These components have a relation with the local linear strain tensor components $\varepsilon_{ij}$ for the materials that obey Hook's law as:

$$t_{ij}(x) = \lambda \varepsilon_{ss}(x) \delta_{ij} + 2G \varepsilon_{ij}(x) \qquad (2)$$

$T$ is the ratio between a characteristic internal length $a$ and characteristic external $l$ length, and $e_o$is a constant which depends on the material and it has to be obtained experimentally or by matching dispersion curves of plane waves with those of atomic-lattice dynamics. $T$ is given by

$$T = \frac{e_o a}{l} \qquad (3)$$

The integral form of the relation given by Eq. (1) can be represented as a differential form as

$$[1 - (e_o a)^2 \nabla^2] \sigma_{ij} = [1 - (Tl)^2 \nabla^2] \sigma_{ij} = E\varepsilon(x) = t_{ij} \qquad (4)$$

### 2.2. Nonlocal Euler-Bernoulli beam equations

The displacements for a beam with length L along its axial direction and its vertical directions are:

$$u_1 = u(x,t) - z\frac{\partial w}{\partial x} \quad u_2 = w(x,t) \quad u_3 = 0 \qquad (5)$$

Where $u$ and $w$ are displacements of the beam along the axial and the transvers directions respectively and there is not any motion along third direction (i.e. $u_3 = 0$). Strain in x direction (axial) is given as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\,\frac{\partial w^2}{\partial x^2} \tag{6}$$

Equations of motion for the beam in axial and transverse directions where rotary inertia is neglected will be

$$\frac{\partial P}{\partial x} + Q(x,t) = \rho A\,\frac{\partial u^2}{\partial t^2} \tag{7}$$

$$\frac{\partial M}{\partial x} + f(x,t) = \rho A\,\frac{\partial w^2}{\partial t^2} \tag{8}$$

where $P$ is the axial force, $Q$ is the horizontal distributed force along the axial direction, $M$ is the resultant bending moment, $f$ is the vertical distributed force, $\rho$ is the density, and A is the cross-sectional area of the beam. Where I is the second moment of inertia and $V$ is the shear force. Where $P$, $M$, $V$, and $I$ are defined as

$$P = \int_A \sigma_{xx}\,dA \qquad M = \int_A -\sigma_{xx}\,z\,dA$$

$$V = \int_A \sigma_{xy}\,dA \qquad I = \int_A z^2 dA \tag{9}$$

According to Reddy [16] and Reddy and Pang [3], the nonlocal form of axial force, the bending moment and the shear force can be written as:

$$P(x) = EA\frac{\partial u}{\partial x} + (e_o a)^2\left[\frac{\partial}{\partial x}\left(\rho A\,\frac{\partial^2 u}{\partial t^2}\right) - \frac{\partial Q}{\partial x}\right]$$

$$M(x) = EI\frac{\partial^2 w}{\partial x^2} + (e_o a)^2\left(\rho A\,\frac{\partial^2 w}{\partial t^2} - f\right)$$

$$V(x) = -EI\frac{\partial^3 w}{\partial x^3} + (e_o a)^2\left[\frac{\partial}{\partial x}\left(\rho A\,\frac{\partial^2 w}{\partial t^2}\right) - \frac{\partial f}{\partial x}\right] \tag{10}$$

Equations of motion of the nonlocal nanobeam for the axial and the lateral displacements according to Reddy [16] and Reddy and Pang [3], are respectively as

$$EA\frac{\partial^2 u}{\partial x^2} + Q - (e_o a)^2\frac{\partial^2 Q}{\partial x^2} = \rho A\,\frac{\partial^2 u}{\partial t^2} \tag{11}$$

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\left[\frac{\partial^2 w}{\partial t^2} - (e_o a)^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial t^2}\right)\right] = f - (e_o a)^2\frac{\partial^2 f}{\partial x^2} \tag{12}$$

For the case of free lateral vibration, all of the external forces must be zero, so Eq. (12) will be changed and used for lateral vibration as

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\left[\frac{\partial^2 w}{\partial t^2} - (e_o a)^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial t^2}\right)\right] = 0 \tag{13}$$

The well-known separation method will be used to solve the above differential equation as

$$w(x,t) = W(x)T(t) \tag{14}$$

Let's assume $c^2 = \frac{\rho A}{EI}$ and ω is the natural frequency of non-cracked beam, then substituting Eq. (14) in Eq. (13) gives

$$\frac{\partial^4 W(x)}{\partial x^4} * \frac{1}{c^2\left[W - (e_o a)^2\frac{\partial^2 W(x)}{\partial x^2}\right]} = \omega^2 \tag{15}$$

Using following dimensionless variables and constants given by

$$\varsigma = \frac{x}{L} \qquad \mu = \frac{e_o a}{L} \qquad \lambda^4 = c^2\omega^2 L^4 = \frac{\rho A\,L^4}{EI}\omega^2$$

$$\overline{W} = \frac{W}{L} \tag{16}$$

Substituting Eq. (16) in Eq. (15) leads to reform Eq. (15) to a spatial equation as

$$\overline{W}^{IV} + \lambda^4(\mu^2\,\overline{W}'' - \overline{W}) = 0 \tag{17}$$

Where $(.)'$ is the derivative with respect to ς. We assume $\overline{W} = me^{st}$ to solve the above differential equation and find its roots as

$$s^4 + \lambda^4(\mu^2 s^2 - 1) = 0 \tag{18}$$

The roots will be as following

$$s_1 = -\beta_1 \qquad s_2 = \beta_1 \qquad s_3 = i\beta_2$$

$$s_4 = -i\beta_2 \tag{19}$$

The general solution of Eq. (17) by using Eq. (19) will be as

$$\overline{W}(\varsigma) = A_1 e^{-i\beta_2 x} + A_2 e^{i\beta_2 x} + A_3 e^{-\beta_1 x} + A_4 e^{\beta_1 x}$$

$$\overline{W}(\varsigma) = C_1\sinh(\beta_1\varsigma) + C_2\cosh(\beta_1\varsigma) + C_3\sin(\beta_2\varsigma) + C_4\cos(\beta_2\varsigma) \tag{20}$$

Where

$$\beta_1 = \lambda^2\,\mu\sqrt{\frac{\sqrt{1+4/\mu^4\lambda^4} - 1}{2}}$$

$$\beta_2 = \lambda^2\,\mu\sqrt{\frac{\sqrt{1+4/\mu^4\lambda^4} + 1}{2}} \tag{21}$$

When the lateral dimensionless displacement is obtained from Eq. (20), the bending slope, the dimensionless bending moment, and the shear force can be obtained respectively form Eq. (10) as

$$\theta(\varsigma) = \overline{W}'(\varsigma)$$

$$\overline{M}(\varsigma) = \frac{M(\varsigma)\,L}{EI} = \overline{W}''(\varsigma) + \mu^2\lambda^4\,\overline{W}(\varsigma)$$

$$\overline{V}(\varsigma) = \frac{V(\varsigma)\,L^2}{EI} = \overline{W}'''(\varsigma) + \mu^2\lambda^4\,\overline{W}'(\varsigma) \tag{22}$$

Constants $C_1$, $C_2$, $C_3$ and $C_4$ in Eq. (20) can be determined through the boundary conditions.

### 2.3. Nonlocal cracked Euler-Bernoulli beam equations

In this case, it is assumed that a beam has one open edge crack of length d located at a distance $\overline{L}$ from the left end and b= $\overline{L}/L$ (b is dimensionless crack distance from the left end of the beam). For the case of the cracked nanobeam as shown Fig. 1, the method which was used by Loya et al. [29], and has been extended by J. Loya et al. [22]. Now it is being used in this paper.

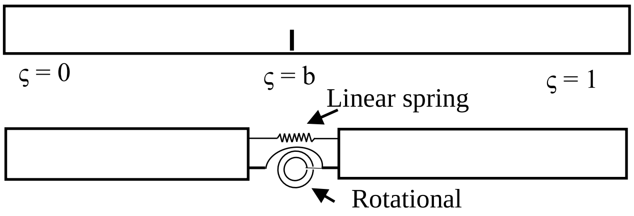



Fig.1 The model of the cracked beam

Crack induces more flexibility to the beam and reduces the stiffness of the beam, therefore, a crack can be modeled as a linear and a rotational spring while the crack induces an additional strain energy to the beam. According to Loya et al. [22], the additional strain energy due to the crack is as

$$\Delta\mathcal{U}_c = \frac{1}{2}\, M\, \Delta\theta + \frac{1}{2}\, P\, \Delta u \tag{23}$$

Where $\Delta\theta$ and $\Delta u$ are the angle of rotation of the rotational spring and the axial displacement of the linear spring respectively. In this work, because there is not any axial force acting on the beam, the amount of $\Delta u$ will be zero. Thus, there is only a rotational spring, and parameter $\Delta\theta$ is given by

$$\Delta\theta = k_{MM}\, \frac{\partial^2 w}{\partial x^2} + k_{MV}\, \frac{\partial u}{\partial x} \tag{24}$$

The crossover flexibility constant $k_{MV}$ is neglected because it is small enough. Now the slope increment $\Delta\theta$ will be rewritten in the form of dimensionless

$$\Delta\theta = \frac{k_{MM}}{L}\, \frac{\partial^2 \bar{W}(\varsigma)}{\partial \varsigma^2}\Bigg|_{\varsigma=b} = K\, \frac{\partial^2 \bar{W}(\varsigma)}{\partial \varsigma^2}\Bigg|_{\varsigma=b} = K\, \bar{W}''(b) \tag{25}$$

where $K = \frac{k_{MM}}{L}$ and it is a dimensionless form. For nanobeams, $\Delta\mathcal{U}_c$ has to be obtained from either molecular dynamics or “ab initio studies”.

**2.3.1. First method for cracked nanobeam**

Each crack divides the beam into two parts, so if the number of the cracks is increased, the number of parts will be increased too. Each part has its own equation of motion as

$$\bar{W}_1^{IV} + \Lambda^4(\mu^2\, \bar{W}_1'' - \bar{W}_1) = 0 \quad 0 \le \varsigma \le b$$
$$\bar{W}_2^{IV} + \Lambda^4(\mu^2\, \bar{W}_2'' - \bar{W}_2) = 0 \quad b \le \varsigma \le 1 \tag{26}$$

The above equation shows the beam has only one crack because there are two equations of motion. Parameter $\Lambda$ is the frequency parameter of the cracked beam, and its relation with natural frequency of the cracked nanobeam ($\omega_c$) is written as

$$\Lambda^4 = c^2\, \omega_c^2\, L^4 = \frac{\rho A\, L^4}{EI}\, \omega_c^2 \tag{27}$$

The same process has been taken in order to find the solution for the case of non-cracked beam, is necessary to be exposed to find the general solution for the case of the cracked beam. Thus, the solution for differential Eq. (26) will be as

$$\bar{W}_1(\varsigma) = C_1 \sinh(\beta_s\varsigma) + C_2 \cosh(\beta_s\varsigma) + C_3 \sin(\beta_f\varsigma) + C_4 \cos(\beta_f\varsigma)$$
$$0 \le \varsigma \le b$$
$$\bar{W}_2(\varsigma) = C_5 \sinh(\beta_s\varsigma) + C_6 \cosh(\beta_s\varsigma) + C_7 \sin(\beta_f\varsigma) + C_8 \cos(\beta_f\varsigma)$$
$$b \le \varsigma \le 1 \tag{28}$$

where coefficients $\beta_s$ and $\beta_f$ for the cracked beam are similar to Eq. (21) and are given as

$$\beta_s = \Lambda^2\, \mu \sqrt{\frac{\sqrt{1+4/\mu^4\,\Lambda^4}-1}{2}}$$
$$\beta_f = \Lambda^2\, \mu \sqrt{\frac{\sqrt{1+4/\mu^4\,\Lambda^4}+1}{2}} \tag{29}$$

There are eight unknown constants in Eq. (28), which have to be obtained by exposing the boundary conditions to Eq. (28) and from the following compatibility equations at the crack position.

- Continuity of the vertical displacement

$$\bar{W}_1(\mathrm{b}) = \bar{W}_2(\mathrm{b}) \tag{30}$$

- Jump in Bending slope

$$\Delta\theta = \bar{W}_2'(\mathrm{b}) - \bar{W}_1'(\mathrm{b}) = K\, \bar{W}_1''(\mathrm{b}) \tag{31}$$

- Continuity of the bending moment

$$\bar{W}_1''(\mathrm{b}) + \Lambda^4\mu^2\, \bar{W}_1(\mathrm{b}) = \bar{W}_2''(\mathrm{b}) + \Lambda^4\mu^2\, \bar{W}_2(\mathrm{b}) \tag{32}$$

- Continuity of the shear force

$$\bar{W}_1'''(\mathrm{b}) + \Lambda^4\mu^2\, \bar{W}_1'(\mathrm{b}) = \bar{W}_2'''(\mathrm{b}) + \Lambda^4\mu^2\, \bar{W}_2'(\mathrm{b}) \tag{33}$$

**2.3.2. Second method for cracked beam**

There is another method proposed by Loya et al. [22], in which the number of the constants, for all cases of single cracked and multi-cracked nanobeams, for all types of beam supports and boundary conditions will be only four unknown constants, and this method gives the same results as the last method. These constants are: vertical displacement $W_o$, bending slope $\theta_o$, bending moment $M_o$, and shear force $V_o$ at $\varsigma = 0$.

$\bar{W}_1(\varsigma) = W_o\, j_1(\varsigma) + \theta_o\, j_2(\varsigma) + M_o\, j_3(\varsigma) + V_o\, j_4(\varsigma) \quad 0 \le \varsigma \le b$

$\bar{W}_2(\varsigma) = \bar{W}_1(\varsigma) + \Delta\theta\, j_2(\varsigma - \mathrm{b}) \quad \mathrm{b} \le \varsigma \le 1 \quad (34)$

Now functions $j_i(\varsigma)$ for all cases, according to Loya [22], are given as

$$\mathrm{j}_1(\varsigma) = \cosh(\beta_s\varsigma) + \frac{(\Lambda^4\mu^2+\beta_s{}^2)\,[\cos(\beta_f\varsigma)-\cosh(\beta_s\varsigma)]}{\beta_f{}^2+\beta_s{}^2} \quad (35)$$

$$\mathrm{j}_2(\varsigma) = \frac{\sin(\beta_f\varsigma)}{\beta_f} - \frac{(\beta_f{}^3-\Lambda^4\mu^2\beta_f)\,[\beta_s\sin(\beta_f\varsigma)-\beta_f\sinh(\beta_s\varsigma)]}{\beta_f{}^2\beta_s(\beta_f{}^2+\beta_s{}^2)} \quad (36)$$

$$\mathrm{j}_3(\varsigma) = \frac{\cosh(\beta_s\varsigma)-\cos(\beta_f\varsigma)}{\beta_f{}^2+\beta_s{}^2} \quad (37)$$

$$\mathrm{j}_4(\varsigma) = \frac{-\beta_f\sinh(\beta_s\varsigma)+\beta_s\sin(\beta_f\varsigma)}{\beta_f\beta_s(\beta_f{}^2+\beta_s{}^2)} \quad (38)$$

In this method, for all types of beam supports and boundary conditions, there will be only four constants that two of them could be determined by the boundary conditions at $\varsigma = 0$. The other two constants are determined by the boundary conditions at $\varsigma = 1$. Then a coefficient matrix will be obtained. Determinant of this coefficient matrix will be set equal to zero and a new equation with only one variable is obtained, then the roots of this equation will give the frequency parameters of the cracked nanobeam.

**2.4. Nonlocal double-cracked Euler-Bernoulli beam equations**

In this section, the equations are derived for double-cracked nanobeam using both methods have been mentioned in last section. A general form is presented for a multi-cracked beam. Equations of the both methods are derived for three different types of supports having different boundary conditions. According to Eq. (28) and Eq. (34) the equations for both methods are derived respectively.

First method to obtain the general equations, and consequently the coefficient matrix, as well as the frequency parameters, is as

$$\bar{W}_1(\varsigma) = C_1\sinh(\beta_s\varsigma) + C_2\cosh(\beta_s\varsigma) + C_3\sin(\beta_f\varsigma) + C_4\cos(\beta_f\varsigma) \quad 0 \le \varsigma \le b_1$$

$$\bar{W}_2(\varsigma) = C_5\sinh(\beta_s\varsigma) + C_6\cosh(\beta_s\varsigma) + C_7\sin(\beta_f\varsigma) + C_8\cos(\beta_f\varsigma) \quad b_1 \le \varsigma \le b_2$$

$$\bar{W}_3(\varsigma) = C_9\sinh(\beta_s\varsigma) + C_{10}\cosh(\beta_s\varsigma) + C_{11}\sin(\beta_f\varsigma) + C_{12}\cos(\beta_f\varsigma) \quad b_2 \le \varsigma \le 1 \quad (39)$$

The boundary conditions for simply supported beam are, as

First B.C.:

$\varsigma = 0 \rightarrow \bar{W}_1(0) = 0,$

$\bar{M}_1(0) = \bar{W}_1''(0) + \mu^2\,\lambda^4\,\bar{W}_1(0) = 0 \quad (40)$

Second B.C.:

$\varsigma = 1 \rightarrow \bar{W}_3(1) = 0,$

$\bar{M}_3(1) = \bar{W}_3''(1) + \mu^2\,\lambda^4\,\bar{W}_3(1) = 0 \quad (41)$

The boundary conditions for clamped-clamped beam are, as

First B.C.:

$\varsigma = 0 \rightarrow \bar{W}_1(0) = 0, \quad \bar{W}_1'(0) = 0 \quad (42)$

Second B.C.:

$\varsigma = 1 \rightarrow \bar{W}_3(1) = 0, \quad \bar{W}_3'(1) = 0 \quad (43)$

The boundary conditions for cantilever beam are, as

First B.C.:

$\varsigma = 0 \rightarrow \bar{W}_1(0) = 0, \quad \bar{W}_1'(0) = 0 \quad (44)$

Second B.C.:

$\varsigma = 1 \rightarrow \bar{M}_3(1) = \bar{W}_3''(1) + \mu^2\,\lambda^4\,\bar{W}_3(1) = 0,$

$\bar{V}_3(1) = \bar{W}_3'''(1) + \mu^2\,\lambda^4\,\bar{W}_3'(1) = 0 \quad (45)$

For all three different types of the beams mentioned above, the following conditions will be the same.

Continuity of the vertical displacements:

$\varsigma = b_1 \rightarrow \bar{W}_1(b_1) = \bar{W}_2(b_1) \quad (46)$

$\varsigma = b_2 \rightarrow \bar{W}_2(b_2) = \bar{W}_3(b_2) \quad (47)$

Jump in Bending slopes:

$\varsigma = b_1 \rightarrow \Delta\theta_1 = \bar{W}_2'(b_1) - \bar{W}_1'(b_1) = K_1\,\bar{W}_1''(b_1) \quad (48)$

$\varsigma = b_2 \rightarrow \Delta\theta_2 = \bar{W}_3'(b_2) - \bar{W}_2'(b_2) = K_2\,\bar{W}_2''(b_2) \quad (49)$

Continuity of the bending moments:

$\varsigma = b_1 \rightarrow \bar{W}_1''(b_1) + \Lambda^4\mu^2\,\bar{W}_1(b_1) = \bar{W}_2''(b_1) + \Lambda^4\mu^2\,\bar{W}_2(b_1)$

(50)

$\varsigma = b_2 \rightarrow \bar{W}_2''(b_2) + \Lambda^4\mu^2\,\bar{W}_2(b_2) = \bar{W}_3''(b_2) + \Lambda^4\mu^2\,\bar{W}_3(b_2)$

(51)

Continuity of the shear forces:

$\varsigma = b_1 \rightarrow \bar{W}_1'''(b_1) + \Lambda^4\mu^2\,\bar{W}_1'(b_1) = \bar{W}_2'''(b_1) + \Lambda^4\mu^2\,\bar{W}_2'(b_1)$

(52)

$\varsigma = b_2 \rightarrow \bar{W}_2'''(b_2) + \Lambda^4\mu^2\,\bar{W}_2'(b_2) = \bar{W}_3'''(b_2) + \Lambda^4\mu^2\,\bar{W}_3'(b_2)$

(53)

There are twelve equations and twelve unknown constants for simply supported and clamped-clamped nanobeams. Two of these constants are zero, thus only ten constants remain. Finally, ten equations will be obtained. As for the clamped-free (i.e. cantilever) nanobeam, none of the constants are zero but two of them are related to each other, that is the reason why the number of unknown constants will be reduced to ten and there will be a coefficient matrix of $10 \times 10$ and its determinant can give the frequencies of the double-cracked nanobeam. For the case of more than two cracks, all of the procedures are the same but, only the number of unknown constants will be increased according to the number of the cracks. If the crack severities are different, then it is necessary to write them into the equations by different names. For example, in the above equations $K_1$ and $K_2$ are independent from another, and each of them is assigned to a particular crack as shown in Fig. 2. The expressions $b_1$ and $b_2$ are positions of the first and the second crack respectively

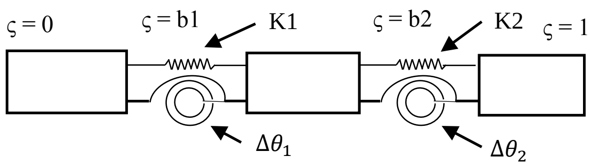



Fig. 2. Doubled-cracked beam

The second method will give simpler and faster calculations in contrast to the first method that was mentioned above. The second method to obtain the general equations, and consequently coefficient matrix, as well as the frequency parameters, is as

$$\bar{W}_1(\varsigma) = W_o\,j_1(\varsigma) + \theta_o\,j_2(\varsigma) + M_o\,j_3(\varsigma) + V_o\,j_4(\varsigma) \quad 0 \le \varsigma \le b_1$$
$$\bar{W}_2(\varsigma) = \bar{W}_1(\varsigma) + \Delta\theta_1\,j_2(\varsigma - b_1) \quad b_1 \le \varsigma \le b_2$$
$$\bar{W}_3(\varsigma) = \bar{W}_2(\varsigma) + \Delta\theta_2\,j_2(\varsigma - b_2) \quad b_2 \le \varsigma \le 1 \quad (54)$$

The first boundary condition for simply supported beam is as

$$\varsigma = 0 \rightarrow W_o = 0,\ and\ M_o = 0 \rightarrow \bar{W}_1(\varsigma) = \theta_o\,j_2(\varsigma) + V_o\,j_4(\varsigma) \quad (55)$$

First boundary condition for clamped-clamped beam is as

$$\varsigma = 0 \rightarrow W_o = 0,\ and\ \theta_o = 0 \rightarrow \bar{W}_1(\varsigma) = M_o\,j_3(\varsigma) + V_o\,j_4(\varsigma) \quad (56)$$

First boundary condition for cantilever beam is as

$$\varsigma = 0 \rightarrow W_o = 0,\ and\ \theta_o = 0 \rightarrow \bar{W}_1(\varsigma) = M_o\,j_3(\varsigma) + V_o\,j_4(\varsigma) \quad (57)$$

As shown in Eqns. (55), (56) and (57), two of the unknown constants out of four constants will be determined by the type of support at $\varsigma = 0$ and two other unknown constants will be obtained by a system of two equations from the boundary condition at $\varsigma = 1$. The second method is better to be used because in all of the cases such as non-cracked, single –cracked, double cracked, and more than two cracks, the coefficient matrix will be $2 \times 2$. The determinant of this coefficient matrix sometimes will be a very long formula that has to be solved numerically to obtain its roots.

**2.4. Nonlocal triple-cracked Euler-Bernoulli beam equations**

This case is similar to the case of the double-cracked nanobeam. It is only needed to expand the equations of the double-cracked nanobeams to the triple-cracked nanobeams as

$$\bar{W}_1(\varsigma) = W_o\,j_1(\varsigma) + \theta_o\,j_2(\varsigma) + M_o\,j_3(\varsigma) + V_o\,j_4(\varsigma) \quad 0 \le \varsigma \le b_1$$
$$\bar{W}_2(\varsigma) = \bar{W}_1(\varsigma) + \Delta\theta_1\,j_2(\varsigma - b_1) \quad b_1 \le \varsigma \le b_2$$
$$\bar{W}_3(\varsigma) = \bar{W}_2(\varsigma) + \Delta\theta_2\,j_2(\varsigma - b_2) \quad b_2 \le \varsigma \le b_3$$
$$\bar{W}_4(\varsigma) = \bar{W}_3(\varsigma) + \Delta\theta_3\,j_2(\varsigma - b_3) \quad b_3 \le \varsigma \le 1$$
$$\Delta\theta_3 = K_3\,\bar{W}_3''(b_3) \quad (58)$$

**3. Results and discussion**

## 3.1. Simply supported beam

In this paper the simply supported beam is analyzed for the cases of double-cracked and triple-cracked. When the cracks are introduced to the beam, the natural frequencies will be decreased and as much as the crack severities are increased the natural frequencies become smaller, and there is an exception in this expression. The exception is the crack location. When the crack location locates on a node of a certain mode of vibration, that mode will not experience any changes by the presence of the crack and increasing the crack severity, consequently there will not be any changes in the frequency of the same mode. This is due to the fact that the amount of the bending slope at both sides of the point on which the crack is located, will be the same, and there will not be any changes in the bending slope and there will not be any jumps in the bending slope. According to Eq. (31), the crack can affect the beam when there is a jump in the bending slope. When both angles at both sides of a point are equal to each other, so amount of $\Delta\theta$ will be zero and this causes the crack effect to be canceled at a mode of vibration in which the crack and one of the nodes of this mode have the same location.

Another factor, which plays an important role in decreasing the natural frequencies, is the size effect introduced by scale effect parameter $\mu$. As $\mu$ is increased, the natural frequencies will be reduced. The results that have already been obtained for simply supported beam as non-cracked beam by Lu et al. [12], are calculated again and completely coincide, then, they are used to be compared with the cases of the double cracked and the triple-cracked. Fig.3 shows the first four frequencies of the non-cracked simply supported beam (i.e. $K_1 = K_2 = K_3 = 0$), and starting from the first mode, the successive odd and even vibration modes approach each other and are suppressed with the increase of $\mu$ .When the number of the cracks is increased, the frequencies of all of the modes will be decreased, except the cases in which one or more cracks are located on the nodes of the certain modes. The results of the double-cracked simply supported beams are tabulated in Table 1.a and b. and are shown in Fig. 4 (a, b, c and d), in which both of the crack severities are changed in accordance with one another. However, it will be possible that each crack severity differs from the other crack severities that is the reason why the crack severity for each crack is named by a different expression such as $K_1$and $K_2$. As for the case of double-cracked, the fourth frequency remains constant while the crack severities are changed, this is because both points $b_1 = 0.25$ and $b_2 = 0.5$ are the nodes of the fourth mode. The same phenomenon occurs for the case of triple-cracked, where all three cracks are located at the nodes of mode four. The results of the triple-cracked simply supported beams are tabulated in Table 2.a and b. and are shown in Fig. 5 (a, b, c and d). The fourth mode remains constant while the crack severities are increased because all cracks are located on the nodes of the fourth mode as shown in Fig. 6. As the scale effect parameter is increased the frequencies of all modes are decreased. The highest amount of decreasing of the frequencies of all modes for any amount of the nonlocal parameter, is observed when the third crack is introduced to the beam.

Table 1. a. and b. Frequencies of double-cracked simply supported beam with different nonlocal parameter $\mu$ and crack severities $K_1$ and $K_2$. Crack positions $\varsigma = 0.25$ and $\varsigma = 0.5$.

| $\mu = 0$ | | | | |
|---|---|---|---|---|
| $\Lambda$ | $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 1 | 3.1416 | 3.0044 | 2.6226 | 1.9256 |
| 2 | 6.2832 | 6.1007 | 5.5522 | 4.2553 |
| 3 | 9.4248 | 9.0315 | 8.1721 | 7.4180 |
| 4 | 12.5664 | 12.5664 | 12.5664 | 12.5664 |
| $\mu = 0.4$ | | | | |
| $\Lambda$ | $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 1 | 2.4790 | 2.3694 | 2.0584 | 1.4983 |
| 2 | 3.8204 | 3.7015 | 3.2600 | 2.3268 |
| 3 | 4.7722 | 4.5563 | 4.1199 | 3.8918 |
| 4 | 5.5509 | 5.5509 | 5.5509 | 5.5509 |

a.

| $\mu = 0.2$ | | | |
|---|---|---|---|
| $K_1 = 0$<br>$K_2 = 0$ | $K_1 = 0.065$<br>$K_2 = 0.065$ | $K_1 = 0.35$<br>$K_2 = 0.35$ | $K_1 = 2$<br>$K_2 = 2$ |
| 2.8908 | 2.7639 | 2.4074 | 1.7604 |
| 4.9581 | 4.8071 | 4.2799 | 3.1046 |
| 6.4520 | 6.1629 | 5.5422 | 5.1663 |
| 7.6407 | 7.6407 | 7.6407 | 7.6407 |
| $\mu = 0.6$ | | | |
| $K_1 = 0$<br>$K_2 = 0$ | $K_1 = 0.065$<br>$K_2 = 0.065$ | $K_1 = 0.35$<br>$K_2 = 0.35$ | $K_1 = 2$<br>$K_2 = 2$ |
| 2.1507 | 2.0552 | 1.7830 | 1.2949 |
| 3.1815 | 3.0820 | 2.7056 | 1.9236 |
| 3.9329 | 3.7548 | 3.4032 | 3.2282 |
| 4.5565 | 4.5565 | 4.5565 | 4.5565 |

b.

Table 2.a and b. Frequencies of a triple-cracked simply supported beam with different nonlocal parameter $\mu$ with three similar cracks of severity $K$ at $\varsigma = 0.25$, $\varsigma = 0.5$, and $\varsigma = 0.75$ .

| $\mu = 0$ | | | | |
|---|---|---|---|---|
| $\Lambda$ | $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 1 | 3.1416 | 2.9652 | 2.5239 | 1.8136 |
| 2 | 6.2832 | 5.9295 | 5.0418 | 3.6171 |
| 3 | 9.4248 | 8.8807 | 7.4775 | 5.2977 |
| 4 | 12.5664 | 12.5664 | 12.5664 | 12.5664 |
| $\mu = 0.4$ | | | | |
| $\Lambda$ | $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 1 | 2.4790 | 2.3390 | 1.9862 | 1.4221 |
| 2 | 3.8204 | 3.5974 | 3.0148 | 2.1232 |
| 3 | 4.7722 | 4.4659 | 3.6264 | 2.4910 |
| 4 | 5.5509 | 5.5509 | 5.5509 | 5.5509 |

a.

| $\mu = 0.2$ | | | |
|---|---|---|---|
| $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 2.8908 | 2.7281 | 2.3195 | 1.6640 |
| 4.9581 | 4.6719 | 3.9320 | 2.7828 |
| 6.4520 | 6.0459 | 4.9386 | 3.4051 |
| 7.6407 | 7.6407 | 7.6407 | 7.6407 |
| $\mu = 0.6$ | | | |
| $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 2.1507 | 2.0290 | 1.7218 | 1.2317 |
| 3.1815 | 2.9953 | 2.5071 | 1.7633 |
| 3.9329 | 3.6793 | 2.9837 | 2.0478 |
| 4.5565 | 4.5565 | 4.5565 | 4.5565 |

b.

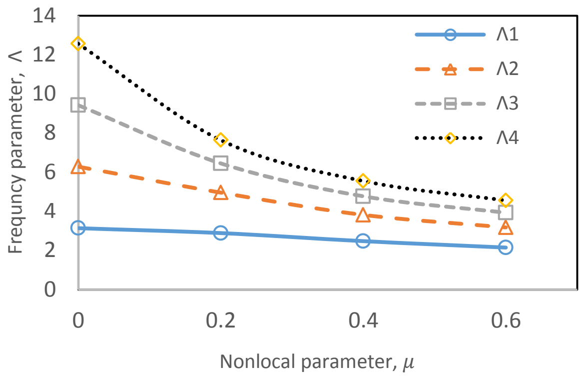
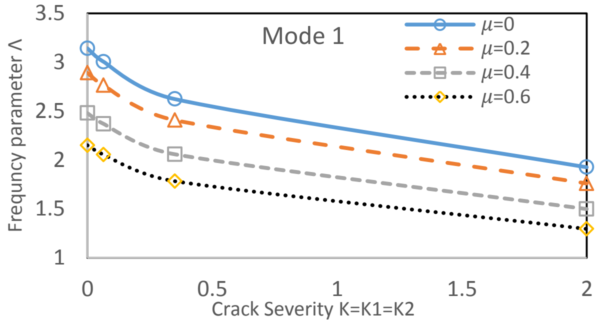
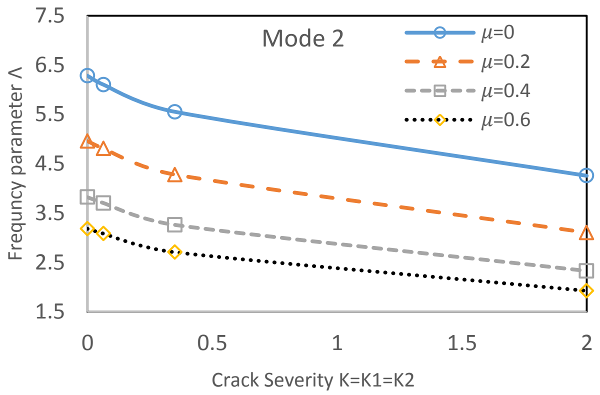


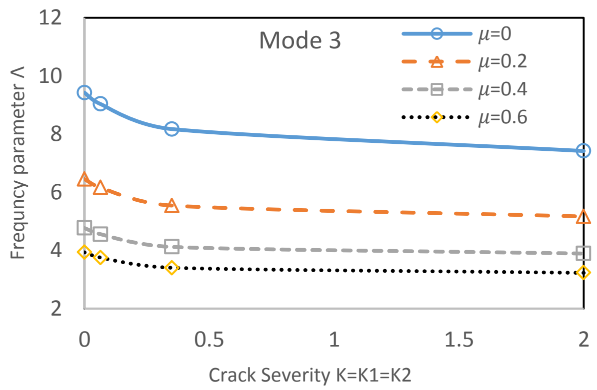

Fig. 3. Change of four eigenvalues of non-cracked simply supported beam versus nonlocal parameter $\mu$.



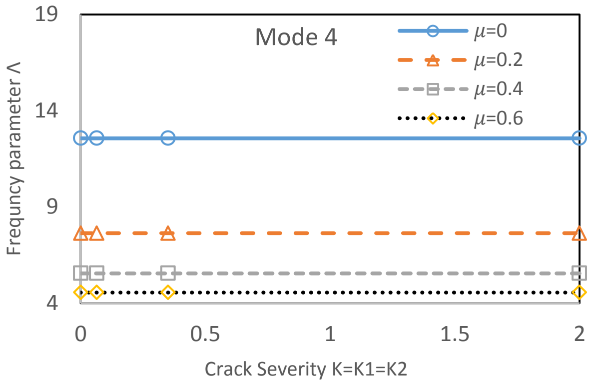

a.




b.

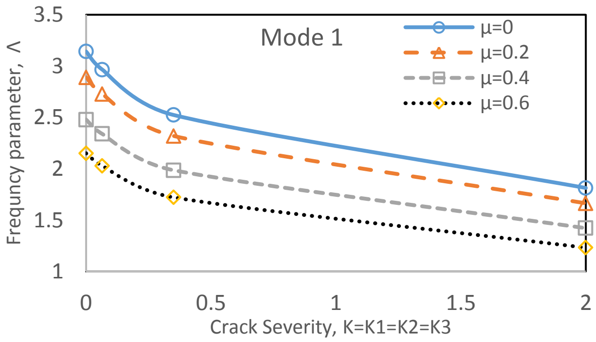



c.

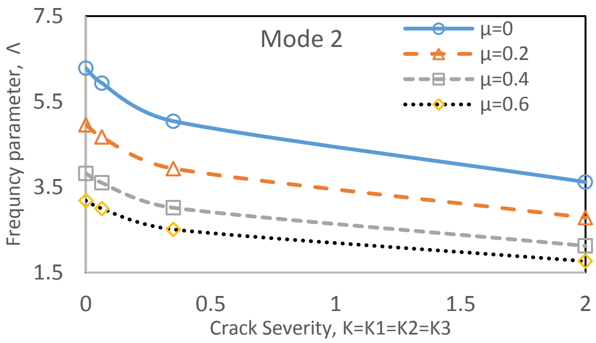



d.

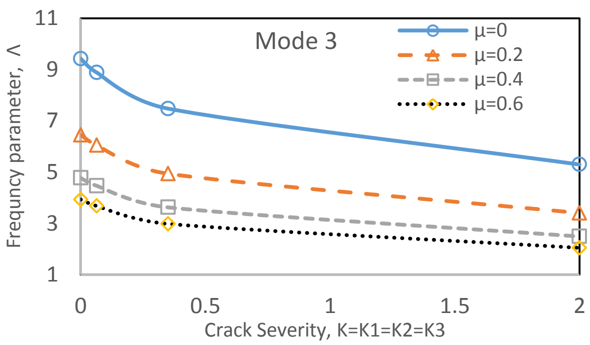
Fig. 4. Frequencies of a double-cracked simply supported beam with crack positions $\varsigma = 0.25$ and $\varsigma = 0.5$.



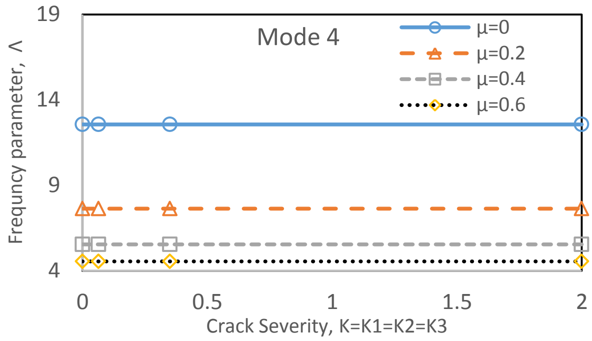

a.




b.




c.




d.

Fig. 5. Frequencies a triple-cracked simply supported beam with crack positions $\varsigma = 0.25$, $\varsigma = 0.5$, and $\varsigma = 0.75$.

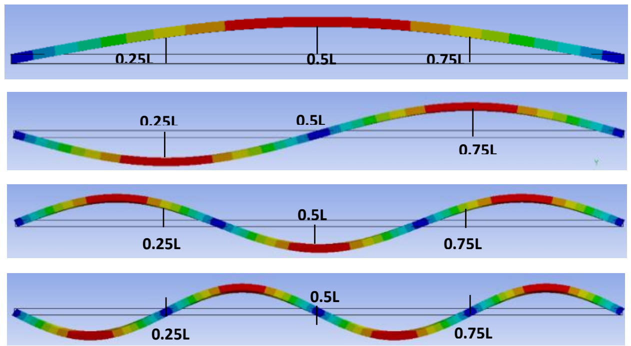



Fig.6. Mode shapes of simply supported beam and position of the nodes.

### 3.2. Clamped-Clamped beam

Non-cracked simply supported beam frequencies are obtained for different nonlocal parameter values. (i.e. $K_1 = K_2 = K_3 = 0$) and the non-cracked case results are shown in Fig. 7. The first four frequency parameters of the clamped-clamped beam are presented in Table 3 (a and b) and in Fig. 8 (a, b, c, and d) for the double-cracked beams and in Table 4 (a and b) and graphically in Fig. 9 for the triple-cracked beams. When the nonlocal parameter is increased the frequencies of all modes are decreased. When the clamped-clamped beam has two cracks at $b_1 = 0.25$ and $b_2 = 0.5$, the frequencies will be decreased more in contrast to the non-cracked beam, and this is due to the presence of the more flexibility in the beam as shown in Fig. 8. In this case, the frequencies of all modes are decreased by increasing the crack severities of the cracks and in fact only the crack severity $K_1$ of the first crack is the reason of decreasing the second and the fourth frequencies, and $K_2$ which belongs to the second crack, does not have any effect on decreasing them because the second crack is located on the nodes of the second and the fourth modes and the amount of $\Delta\theta_2 = 0$ as shown in Fig. 10. Increasing the nonlocal scale effect parameter is another factor for decreasing the frequencies of the double-cracked beam. Third crack decreases all mode frequencies and the lowest amount of the frequencies belongs to the case of the triple-cracked beam as shown in Fig. 9 (a, b, c, and d). In this case only the first and the third cracks are the reason of decreasing the second and the fourth modes. The third crack location is not the node for any modes.

Table 3. Frequencies of a double-cracked clamped-clamped beam with different nonlocal parameter $\mu$ and crack severities $K_1$ and $K_2$. Crack positions $\varsigma = 0.25$ and $\varsigma = 0.5$.

| $\mu = 0$ | | | | |
|---|---|---|---|---|
| $\Lambda$ | $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 1 | 4.7300 | 4.6276 | 4.3531 | 3.8350 |
| 2 | 7.8532 | 7.6974 | 7.2501 | 6.5217 |
| 3 | 10.9956 | 10.4769 | 9.5173 | 8.8476 |
| 4 | 14.1372 | 14.0909 | 14.0119 | 13.9498 |
| $\mu = 0.4$ | | | | |
| $\Lambda$ | $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 1 | 3.5923 | 3.4947 | 3.1864 | 2.5767 |
| 2 | 4.5978 | 4.4938 | 4.0780 | 3.6656 |
| 3 | 5.4738 | 5.1691 | 4.7791 | 4.6427 |
| 4 | 6.1504 | 6.0837 | 5.8674 | 5.6307 |

a.

| $\mu = 0.2$ | | | |
|---|---|---|---|
| $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 4.2766 | 4.1735 | 3.8721 | 3.2575 |
| 6.0352 | 5.9028 | 5.3861 | 4.7371 |
| 7.3840 | 6.9713 | 6.3598 | 6.1174 |
| 8.4624 | 8.3863 | 8.1469 | 7.8308 |
| $\mu = 0.6$ | | | |
| $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 3.0837 | 2.9949 | 2.7068 | 2.1586 |
| 3.8165 | 3.7298 | 3.3905 | 3.0745 |
| 4.5231 | 4.2743 | 3.9691 | 3.8669 |
| 5.0505 | 4.9932 | 4.8082 | 4.6154 |

b.

Table 4. frequencies of a triple-cracked clamped-clamped beam with different nonlocal parameter $\mu$ with three similar cracks of severity $K$ at $\varsigma = 0.25$, $\varsigma = 0.5$, and $\varsigma = 0.75$ .

| $\mu = 0$ | | | | |
|---|---|---|---|---|
| $\Lambda$ | $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 1 | 4.7300 | 4.6268 | 4.3497 | 3.6513 |
| 2 | 7.8532 | 7.5618 | 6.9381 | 6.3018 |
| 3 | 10.9956 | 10.2411 | 8.6693 | 7.0230 |
| 4 | 14.1372 | 14.0397 | 13.8556 | 13.7059 |
| $\mu = 0.4$ | | | | |
| $\Lambda$ | $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 1 | 3.5923 | 3.4946 | 3.1619 | 2.3325 |
| 2 | 4.5978 | 4.4193 | 3.9742 | 3.5209 |
| 3 | 5.4738 | 5.0222 | 4.2248 | 3.7754 |
| 4 | 6.1504 | 6.0048 | 5.7356 | 5.5906 |

a.

| $\mu = 0.2$ | | | |
|---|---|---|---|
| $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 4.2766 | 4.1734 | 3.8595 | 3.0030 |
| 6.0352 | 5.8021 | 5.2461 | 4.6762 |
| 7.3840 | 6.7818 | 5.6246 | 4.7921 |
| 8.4624 | 8.2964 | 7.9512 | 7.7323 |
| $\mu = 0.6$ | | | |
| $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 3.0837 | 2.9946 | 2.6795 | 1.9443 |
| 3.8165 | 3.6683 | 3.2940 | 2.9138 |
| 4.5231 | 4.1520 | 3.5236 | 3.1947 |
| 5.0505 | 4.9253 | 4.7018 | 4.5864 |

b.

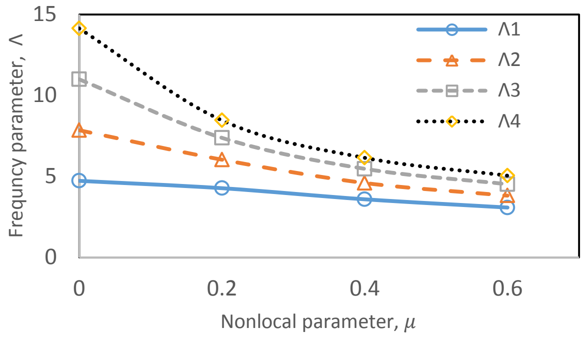
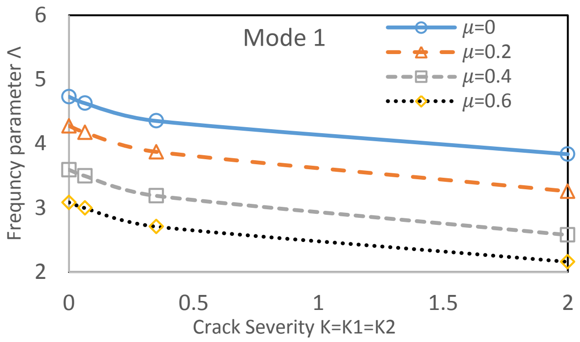
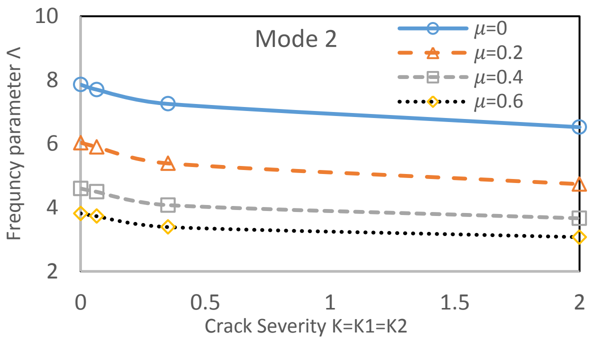


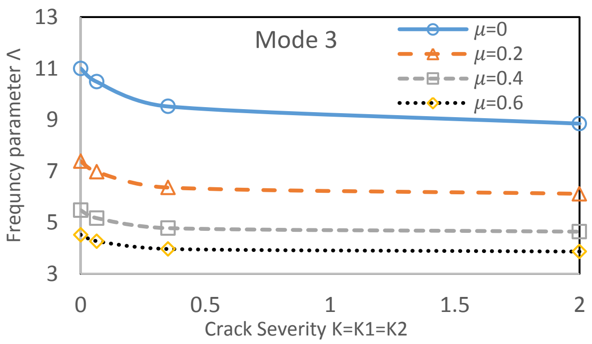

Fig. 7. Change of four eigenvalues of non-cracked clamped-clamped beam versus nonlocal parameter $\mu$.



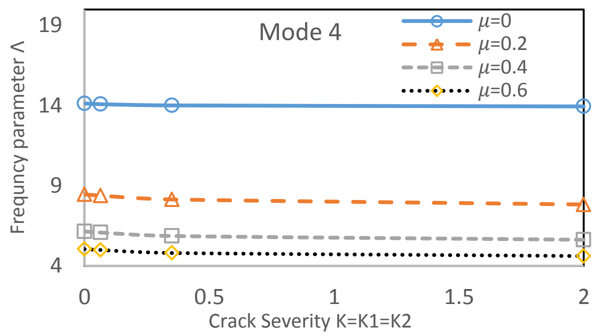

a.




b.

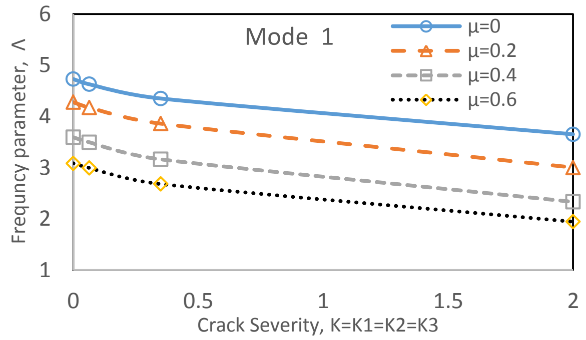



c.

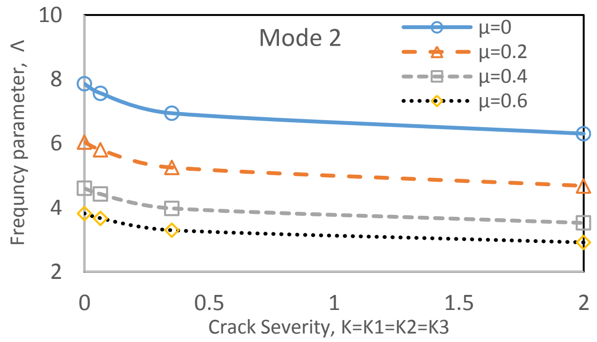



d.

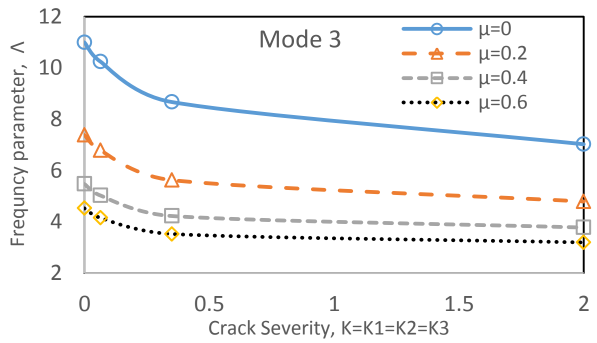
Fig. 8. Frequencies of a double-cracked clamped-clamped beam with crack positions $\varsigma = 0.25$ and $\varsigma = 0.5$.



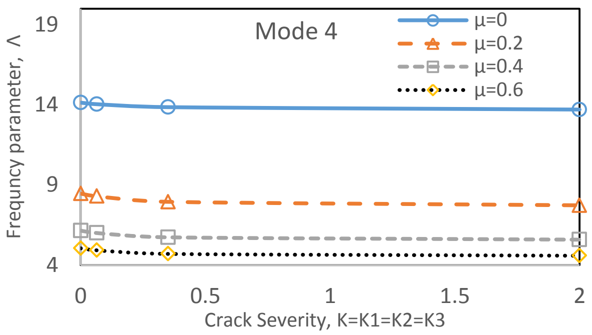

a.




b.




c.




d.

Fig. 9. Frequencies of a triple-cracked clamped-clamped beam with crack positions $\varsigma = 0.25$, $\varsigma = 0.5$, and $\varsigma = 0.75$.

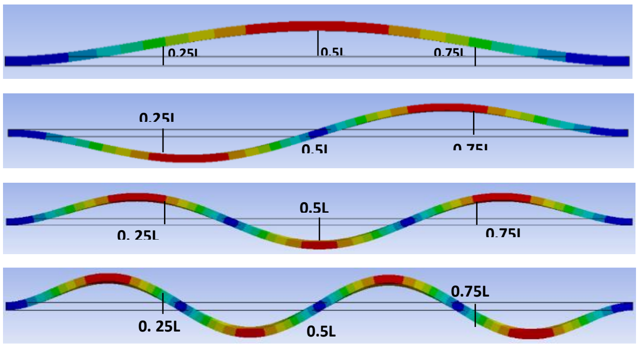



Fig. 10. Mode shapes of clamped-clamped beam and position of the nodes

### 3.3. Cantilever beam

The results of the cantilever beam for the non-cracked case are shown in Fig. 11. Other results of the cantilever are presented in Table 5 and in Fig. 12 (a, b, c, and d) for the double-cracked beams, and in Table 6 and in Fig. 13 (a, b, c, and d) for the triple-cracked beams. General mode shapes of a cantilever beam are shown in Fig. 14. It can be observed from the results that the natural frequencies are very sensitive to the nonlocal parameter in contrast to the simply supported and the clamped-clamped beams, especially for the cases in which the nonlocal parameter ($\mu$) increases. The cantilever beam is not suitable to be used in the design of the resonators in nanoscale. As for the case of non-cracked (i.e. $K_1 = K_2 = K_3 = 0$), only the first frequency increases by an increase in the nonlocal parameter, whereas the others decrease by increasing the parameter $\mu$ and the frequencies approach each other in pairs as shown in Fig. 11. When $\mu > 0.62$, one even cannot find nontrivial real eigenvalues. This means that starting from the first mode, the successive odd and even vibration modes approach each other and are suppressed with the increase of $\mu$. Thus, for the cantilever beam vibration mode, the magnitude of the exponential terms increases dramatically with the nonlocal parameter comparing with the traveling wave terms, and eventually, it restrains the vibration. For the cantilever as the cracks are introduced to the beam, the frequencies of all modes are reduced, and as the crack severities are increased the rate of decreasing the frequencies becomes more. The doubled-cracked cantilever beam results are presented in Table 5 and they are shown in Fig. 12 (a, b, c, and d), and when they are compared with the other cases, it is observed that all of the frequencies are decreased except the fourth mode frequency of case $K_1 = K_2 = 2$ for $\mu = 0.6$, where it is increased. It is one of the abnormal results happened here. Third crack does not have any great effect on the first frequency of the cantilever beam, then the first frequency in both double-cracked and triple-cracked beam have approximately the same value for all the amounts of the nonlocal parameter, but frequencies of the other modes are decreased by introducing the third crack to the beam. There are some abnormal results obtained in the cantilever cases which have not been happened for the simply supported and the clamped-clamped beam cases. The reason is the fact that the higher value of $\mu$ has more complex effect on the frequencies, especially when both $\mu$ and $K$ are increased simultaneously and it is sometimes unpredictable to know what will be happened for a case in which the amount of the nonlocal parameter is high.

Table 5. First four frequency parameters for a double-cracked cantilever beam with different nonlocal parameter $\mu$ and crack severities $K_1$ and $K_2$. Crack positions $\varsigma = 0.25$ and $\varsigma = 0.5$.

| $\mu = 0$ | | | | |
|---|---|---|---|---|
| $\Lambda$ | $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 1 | 1.8751 | 1.8136 | 1.6255 | 1.2287 |
| 2 | 4.6941 | 4.5518 | 4.1572 | 3.2994 |
| 3 | 7.8548 | 7.6994 | 7.2454 | 6.5074 |

| | | | | |
|---|---|---|---|---|
| 4 | 10.9955 | 10.4809 | 9.5378 | 8.8927 |
| $\mu = 0.4$ | | | | |
| $\Lambda$ | $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 1 | 1.9543 | 1.8906 | 1.6909 | 1.2690 |
| 2 | 3.3456 | 3.2155 | 2.8479 | 2.1148 |
| 3 | 4.8370 | 4.7655 | 4.1575 | 3.6429 |
| 4 | 5.2399 | 4.9629 | 4.8385 | 4.8043 |

a.

| $\mu = 0.2$ | | | |
|---|---|---|---|
| $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 1.8919 | 1.8299 | 1.6394 | 1.2374 |
| 4.1924 | 4.0460 | 3.6277 | 2.7481 |
| 6.0674 | 5.9395 | 5.4040 | 4.7359 |
| 7.3617 | 6.9726 | 6.4311 | 6.2394 |
| $\mu = 0.6$ | | | |
| $K_1 = 0$ $K_2 = 0$ | $K_1 = 0.065$ $K_2 = 0.065$ | $K_1 = 0.35$ $K_2 = 0.35$ | $K_1 = 2$ $K_2 = 2$ |
| 2.1989 | 2.1380 | 1.8903 | 1.3706 |
| 2.4809 | 2.3731 | 2.1291 | 1.6387 |
| … | … | 3.5533 | 3.0490 |
| … | … | 3.9588 | 4.0261 |

b.

Table 6. Frequencies of a triple-cracked cantilever beam with different nonlocal parameter $\mu$ with three similar cracks of severity $K$ at $\varsigma = 0.25$, $\varsigma = 0.5$ and $\varsigma = 0.75$ .

| $\mu = 0$ | | | | |
|---|---|---|---|---|
| $\Lambda$ | $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 1 | 1.8751 | 1.8121 | 1.6206 | 1.2214 |
| 2 | 4.6941 | 4.4054 | 3.6597 | 2.5515 |
| 3 | 7.8548 | 6.9997 | 5.2577 | 3.8570 |
| 4 | 10.9955 | 9.9345 | 8.1246 | 6.6601 |
| $\mu = 0.4$ | | | | |
| $\Lambda$ | $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 1 | 1.9543 | 1.8950 | 1.7050 | 1.2886 |
| 2 | 3.3456 | 3.1834 | 2.7516 | 2.0012 |
| 3 | 4.8370 | … | 3.8404 | 2.5818 |
| 4 | 5.2399 | … | 4.0351 | 3.6505 |

a.

| $\mu = 0.2$ | | | |
|---|---|---|---|
| $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 1.8919 | 1.8299 | 1.6392 | 1.2371 |
| 4.1924 | 4.0155 | 3.5370 | 2.6288 |
| 6.0674 | 5.7618 | 4.9197 | 3.4561 |
| 7.3617 | 6.7084 | 5.5042 | 4.7355 |
| $\mu = 0.6$ | | | |
| $K = 0$ | $K = 0.065$ | $K = 0.35$ | $K = 2$ |
| 2.1989 | … | … | … |
| 2.4809 | … | … | … |
| … | … | … | … |
| … | … | … | … |

b.

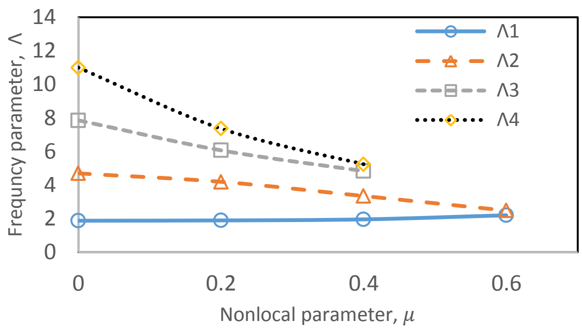
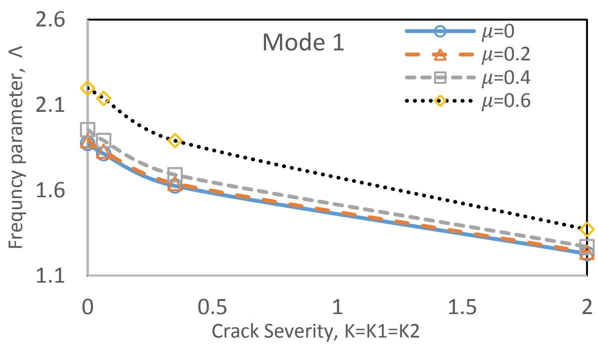
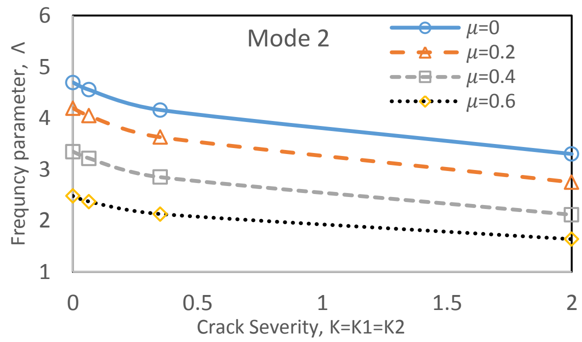


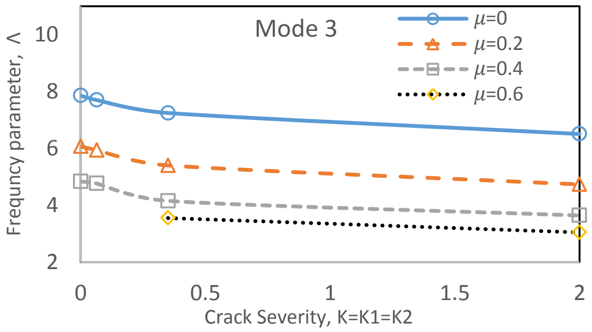

Fig. 11. Change of four eigenvalues of non-cracked cantilever beam versus nonlocal parameter $\mu$



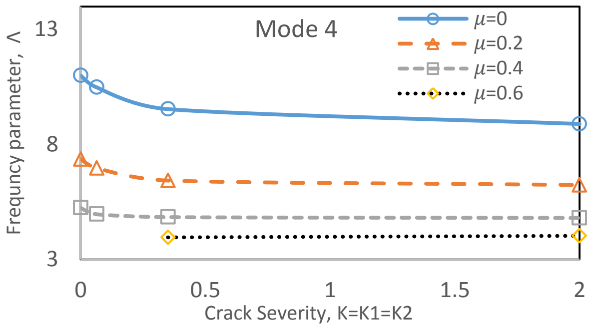

a.




b.

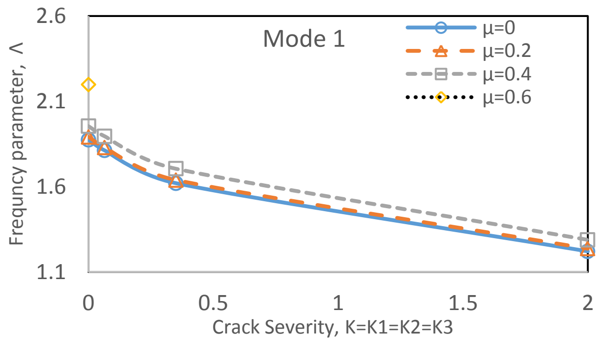



c.

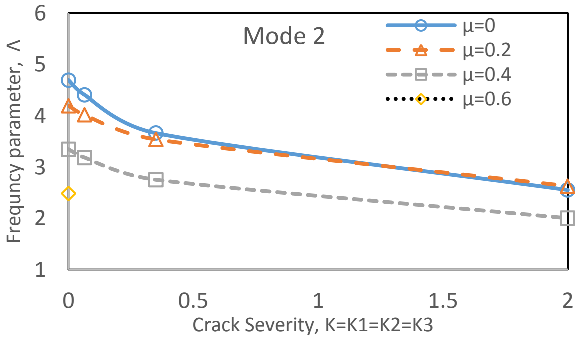



d.

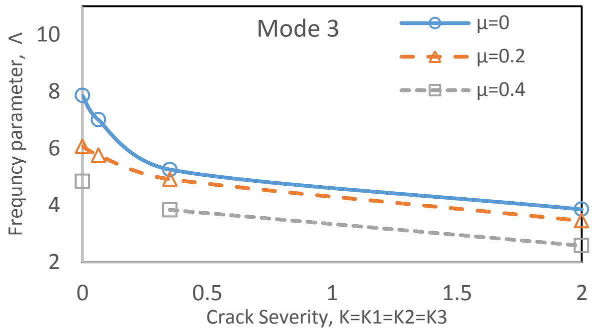
Fig. 12. Frequencies of a double-cracked cantilever beam with crack positions $\varsigma = 0.25$ and $\varsigma = 0.5$.



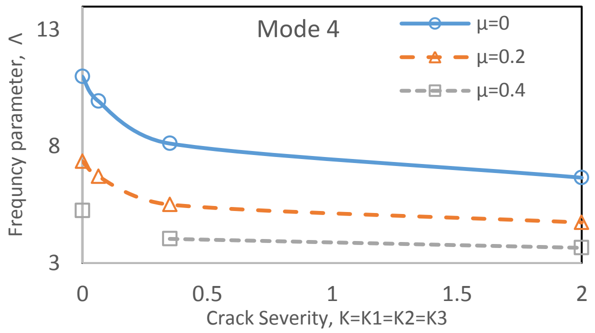

a.




b.




c.




d.

Fig. 13. Frequencies of a triple-cracked cantilever beam with crack positions $\varsigma = 0.25$, $\varsigma = 0.5$, and $\varsigma = 0.75$.

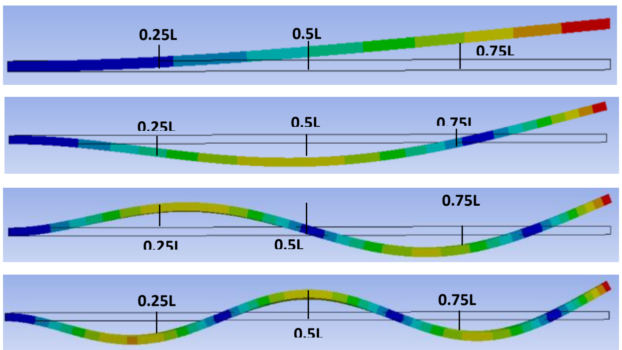



Fig. 14. Mode shapes of cantilever beam and position of the nodes.

## 4. Conclusions

In this paper, free vibration analysis of double-cracked and triple-cracked nanobeams for three different types of beam supports, including simply supported, clamped-clamped, and cantilever is exposed to find the natural frequencies. The crack is modeled as a rotational spring and the value of the crack severities are calculated using molecular dynamics for the nanobeams. The effect of the crack severities, number of the cracks, and the nonlocal parameter are checked in this paper. The nonlocal parameter is considered in the equations and its effect on the frequencies of all studied cases is determined. The Following conclusions have been made based on the results obtained throughout this paper:

- In the cases of the simply supported and the clamped-clamped as the crack severity increases, the frequencies decrease for all values of the nonlocal parameter.
- As the position of the crack gets near the fixed end, the crack effect decreases. As the number of the cracks is increased, there will be a reduction in the frequencies.
- As for the cantilever nanobeam, the results are somehow complicated. When the crack position is closer to the free end, the frequencies of some modes increase and this is a new phenomenon that was observed here.
- For the cantilever nanobeam, the small size effect parameter (nonlocal parameter) has the greatest effect among three cases of support nanobeams.
- The first mode frequency of the cantilever beam increases by an increase in the nonlocal parameter.

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