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RESEARCH PAPER

NONSTANDARD COMPLETION OF NON-COMPLETE METRIC SPACE

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A B S T R A C T:

Our aim in this study is to establishing nonstandard foundations, definitions and theorems for completion a noncomplete metric spaces. We have a lot of space or sets X which agree with all usual properties of complete, except at a small size subset of it. In this paper, by using nonstandard analysis tools founded by A. Robinson and axiomatized by E. Nelson, we try to reformulate the definition of completion corresponding to nonstandard modified metric $\hat{d}$, and to give a nonstandard form to the classical (standard) completion theorem and to use the power of nonstandard tools to overcome the incompetence of those spaces which has deprivation at a small size subset.

KEY WORDS: Nonstandard, infinitesimal, completion, unlimited, infinitely close, non-complete spaces.
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1.INTRODUCTION :

In this paper, we present a nonstandard way for constructing a complete metric space which satisfy all conditions of metric spaces and it is a completion of a non-complete metric space and to overcome the gaps of the classical treatments by introducing a new definitions compatible with our aim for giving a precise and perfect proof for completion of non-complete metric spaces. In mathematical analysis the standard complete metric space it means that every Cauchy sequences in the metric space is converge to a point in it (Macías-Díaz, 2015). Throughout this paper $\mathbb{R}^*$ is the extension of $\mathbb{R}$, includes all real numbers together with nonstandard quantities, and sometimes $\mathbb{R}^*$called it set of hyperreals.

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2. BASIC CONCEPTS

Definition 2.1 (Keisler, 1976)
An element $x \in \mathbb{R}^*$ is
Infinitesimal if $|x| < r$ for all positive real r;
Limited if $|x| < r$ for some real r;
Unlimited if $|x| > r$ for all real r.

Definition 2.2 (Goldbring, 2014).
For $x, y \in \mathbb{R}^*$, we say x and y are infinitely close, written, $x \simeq y$, if $x - y$ is infinitesimal.
Every infinitesimal is limited

Definition 2.3 (Goldblatt, 1998)
A real number x is called ***appreciable*** if it is neither infinite nor infinitesimal.

Definition 2.4 (Keisler, 1976)
Given a hyperreal number $x \in \mathbb{R}^*$, the ***monad*** of x is the set

$$monad(x) = \{y \in \mathbb{R}^*: x \simeq y\}.$$

The ***galaxy*** of x is the set

$$galaxy(x) = \{y \in \mathbb{R}^*: x - y \text{ is limited}\}.$$

Definition 2.5 (Goldbring, 2014)

(1) The set of limited hyperreals is
$$R_{lim} := \{x \in \mathbb{R}^* \mid |x| \leq n \text{ for some } n \in N\}.$$
(2) The set of infinitesimal hyperreals is
$$R_{inf} := \{ x \in \mathbb{R}^* \mid |x| \leq 1/n \text{ for all } n \in N > 0 \}.$$
(3) The set of unlimited hyperreals is
$$R_{unl} := \mathbb{R}^* \setminus R_{lim}.$$

Theorem 2.6 (Existence of Standard Parts) (Goldbring, 2014)
If $r \in R_{lim}$, then there is a unique $s \in \mathbb{R}$ such that $r \simeq s$. We call s the standard part (or shadow) of r and denoted by $st(r)$ or $°r$.

Corollary 2.7 (Keisler, 2011)
Let x and y be a finite
i. $x \simeq y$ if and only if $st(x) = st(y)$.
ii. $x \simeq st(x)$.
iii. If $r \in \mathbb{R}$ then $st(r) = r$.
iv. If $x \leq y$ then $st(x) \leq st(y)$.

3- MAIN RESULTS

Next we give a first results for our modification and extinction of the classical notions about complete, isometry, and dense with respect to nonstandard definitions given in previous. First we start with some new notions about ordering of sequences. All of this modifications and new notions will be necessary for nonstandard construction proof of completion of a non-complete metric space given the last.

Definition 3.18
Let (X, d) be a metric space and let C be the set of all Cauchy sequence in X, the two sequences $x = \{x_n\}, y = \{y_n\}$ *in C are* said to be:
1) In the same order and denoted by $x \simeq^{\Delta} y$, if $x_n \simeq y_n \ \forall n$.
2) Standardly in the same order denoted by $x \simeq_s y$, if $x_n \simeq y_n \ \forall^{st} n$.
3) Unlimitedly in the same order denoted by $x \simeq_{\omega} y$, if $x_n \simeq y_n \ \forall^{unlimited}\, n$.

Definition 3.2
Let (X, d) be a complete metric space and $Y \subseteq X$. Then we say that Y is **nearly complete** if Y is dense in X.

Definition 3.3
Let (X, d) and (Y, ρ) be metric spaces. Then by **nearly isometry** we mean a function $f: X \to Y$ where f is a bijection and for all $x, y \in X$ and
$$\rho(f(x), f(y)) \simeq d(x, y) \forall x, y \in X.$$
In general, a function $i: X \to Y$ is **nearly isometry** we can write a function h if
$$h\big(i(x), i(y)\big) \simeq h(x, y) \qquad \forall x, y \in X.$$

Definition 3.4
Let (X, d) be a matric space. A subset $S \subseteq X$ is said to be **nearly dense** if S can be nearly isometrically embedded in X.

Lemma 3.5
Let (X, d) and (Y, ρ) be two metric spaces such that X is standard. Then a function $f: X \to Y$ is nearly isometry if and only if for all $x, y \in X$ there exists $f(x), f(y) \in f(X)$ such that $°\rho\big(f(x), f(y)\big) = d(x, y)$.

Proof
Let (X, d) and (Y, ρ) be two metric spaces such that $f: X \to Y$ is nearly isometry.
Then f is bijective and for all $x,y \in X, \rho(f(x), f(y)) \simeq d(x, y)$.
Since X is standard then $d(x, y)$ is standard.
Thus, by definition of shadow, we have $°\rho\big(f(x), f(y)\big) = d(x, y)$.
Conversely, assume that X is standard and $f: X \to Y$ be a function such that $°\rho\big(f(x), f(y)\big) = d(x, y) \ \forall x, y \in X$, then $\rho(f(x), f(y)) \simeq d(x, y)$.
Now we have to show that f is bijective.
If $f(x) = f(y)$, then $°\rho\big(f(x), f(y)\big) = 0$.
Hence, $d(x, y) = 0$.
Therefore, $x = y$.
Thus, f is on to one and f is onto by hypothesis and $°\rho\big(f(x), f(y)\big) = d(x, y) \ \forall x, y \in X$.
Therefore, f is bijective.
Hence, f is nearly isometry.

Definition 3.6
Let $\hat{X} = \{ [\lambda]: \alpha_n \to \lambda;\ \alpha_n \in X\}$, X is an arbitrary metric space equipped with metric d. where
$$[\lambda] = \begin{cases} \alpha_n \in C : \ \alpha_n \to \lambda \in X \\ \beta_n \in C : \ \beta_n \to \lambda \notin X \end{cases}$$
Define
$\hat{d}: \hat{X} \to \mathbb{R}$ by: $\hat{d}([\lambda], [\mathfrak{z}]) =$
$$\begin{cases} d(\alpha_n, \beta_n) & if\ n\ is\ unlimited \\ 0 & otherwise \end{cases}$$
Where $\alpha_n, \beta_n \in C \subset X$ such that $\alpha_n \to \lambda, \beta_n \to \mathfrak{z}$, and we define equality among members of $\hat{X}$ by:
$[\lambda] = [\mathfrak{z}] \equiv \lambda \simeq \mathfrak{z} \equiv \alpha_n \simeq \beta_n \ \forall^{unlimited}\, n$.

The following examples are to verify our **Definition 3.6**

Example 3.7

1. Let $X=(0,1]$ and $\lambda=0$, then

$$\hat{X}=\{[0]:\alpha_n\to 0;\ \alpha_n\in X\},$$

And

$$[0]=\begin{cases}\alpha_n\in C:\ \alpha_n\to 0\in X\\ \\ \beta_n\in C:\ \beta_n\to 0\notin X\end{cases}$$

2. Let $X=\mathbb{Q}$ and $\lambda\in\mathbb{R}$, then

$$\hat{X}=\{[\lambda]:\alpha_n\to\lambda;\ \alpha_n\in\mathbb{Q}\},$$

And

$$[\lambda]=\begin{cases}\alpha_n\in C:\ \alpha_n\to\lambda\ \in\mathbb{Q}\\ \\ \beta_n\in C:\ \beta_n\to\ \lambda\notin\mathbb{Q}\end{cases}$$

Definition 3.8

Let $\hat{Y}$ be a subset of $\hat{X}$ such that $\hat{Y}=\{[S_x],x\in X\}$. Then we define S_x to be the $S_x=\{s_n\in C: s_n\simeq x\ \forall^{unlimited}\ n\}$.

Theorem 3.9

$(\hat{X},\hat{d})$ is a metric space.

Proof

We have to show that $\hat{d}$ is define a metric on $\hat{X}$. From **Definition 3.6** we have

$$[\lambda]=[\mathfrak{z}]\equiv\lambda\simeq\mathfrak{z}$$
$$\equiv\alpha_n\simeq\beta_n\equiv\ d(\alpha_n,\beta_n)\simeq 0\ \forall^{unlimited}\ n.$$

Thus,

$$\hat{d}([\lambda],[\mathfrak{z}])=\begin{cases}d(\alpha_n,\beta_n)=\begin{cases}c\ \ n\ unlimited\ \ \alpha_n\not\simeq\beta_n\\ \varepsilon\ \ n\ unlimited\ \ \alpha_n\simeq\beta_n\end{cases}\\ \quad 0 \qquad\qquad\qquad otherwise\end{cases}$$

Where c is an appreciable real and ε is infinitesimal. Before we prove that $(\hat{X},\hat{d})$ is a metric space, we have to prove the $\hat{d}([\lambda],[\mathfrak{z}])$ is really exists for every $\alpha_n,\beta_n\in C\subset$X. we have

$$|d(\alpha_n,\beta_n)-d(\alpha_m,\beta_m)|\le|d(\alpha_n,\beta_n)-d(\alpha_n,\beta_m)|+|d(\alpha_n,\beta_m)-d(\alpha_m,\beta_m)| \qquad (1)$$

Also, for d a metric, we have

$$|d(x,y)-d(y,z)|\le d(x,z)\quad\forall x,y,z\in X \qquad (2)$$

Applying (2) in (1) we obtain

$$|d(\alpha_n,\beta_n)-d(\alpha_m,\beta_m)|\le d(\beta_n,\beta_m)+d(\alpha_n,\alpha_m) \qquad (3)$$

Since both α_n,β_n are Cauchy sequences in X, then

$d(\beta_n,\beta_m)\simeq 0, d(\alpha_n,\alpha_m)\simeq 0\ \forall^{unlimited}\ n,m.$

Equivalently,

$$\beta_n\simeq\beta_m,\alpha_n\simeq\alpha_m\ \forall^{unlimited}n,m \qquad (4)$$

From (3), (4) we get that $|d(\alpha_n,\beta_n)-d(\alpha_m,\beta_m)|\simeq 0\ \forall^{unlimited}\ n,m.$

Equivalently,

$d(\alpha_n,\beta_n)\simeq d(\alpha_m,\beta_m)\ \forall^{unlimited}\ n,m,$

It follows that $\{d(\alpha_n,\beta_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete then $d(\alpha_n,\beta_n)$ is converge to a value in $\mathbb{R}$. That is $d(\alpha_n,\beta_n)$ is exists and since d is metric, then $d(x,y)=0\Leftrightarrow x=y$. Thus, if $x\simeq y\Rightarrow d(x,y)\simeq 0$.

To show that $\hat{d}$ is well defined, we have to prove that if $[\lambda]=[\lambda^*]$ and $[\mathfrak{z}]=[\mathfrak{z}^*]$, then $\hat{d}([\lambda],[\mathfrak{z}])=\hat{d}([\lambda^*],[\mathfrak{z}^*])$ for $\alpha_n\to\lambda,\beta_n\to\mathfrak{z},{\alpha_n}^*\to\lambda^*$ and ${\beta_n}^*\to\mathfrak{z}^*$.

Now, for any positive integer n we have

$$d(\alpha_n,\beta_n)\le d(\alpha_n,{\alpha_n}^*)+d({\alpha_n}^*,\beta_n)$$
$$d(\alpha_n,\beta_n)\le d(\alpha_n,{\alpha_n}^*)+d({\alpha_n}^*,{\beta_n}^*)+d({\beta_n}^*,\beta_n)$$

Thus as $[\lambda]=[\lambda^*]$, then $\lambda\simeq\lambda^*$ and so as $\mathfrak{z}\simeq\mathfrak{z}^*$. Therefore, $\alpha_n\simeq{\alpha_n}^*$ and $\beta_n\simeq{\beta_n}^*$. Then

$d(\alpha_n,{\alpha_n}^*)\simeq 0\ and\ d({\beta_n}^*,\beta_n)\simeq 0.$ Therefore,

$$d(\alpha_n,\beta_n)\lesssim\ d({\alpha_n}^*,{\beta_n}^*) \qquad (5)$$

Similarly,

$$d({\alpha_n}^*,{\beta_n}^*)\lesssim\ d(\alpha_n,\beta_n) \qquad (6)$$

From (5) and (6) we obtain that

$d(\alpha_n,\beta_n)=\ d({\alpha_n}^*,{\beta_n}^*)$

Thus $\hat{d}([\lambda],[\mathfrak{z}])=\hat{d}([\lambda^*],[\mathfrak{z}^*])$.

To show that $\hat{d}$ is a metric on $\hat{X}$.

1) Since d is metric on X then $\hat{d}([\lambda],[\mathfrak{z}]) \gtrsim 0$.
2) Let $[\lambda]=[\mathfrak{z}]$ then $\forall \alpha_n, \beta_n \in C \subseteq X$ $\alpha_n \simeq \lambda \simeq \mathfrak{z} \simeq \beta_n \Leftrightarrow d(\alpha_n,\beta_n) \equiv d(\lambda,\mathfrak{z}) = 0 \Leftrightarrow \hat{d}([\lambda],[\mathfrak{z}]) = 0$.
3) It is clear $\hat{d}([\lambda],[\mathfrak{z}]) = \hat{d}([\mathfrak{z}],[\lambda])$.
4) $\hat{d}([\lambda],[\mathfrak{z}]) \leq \hat{d}([\lambda],[\varphi]) + \hat{d}([\varphi],[\mathfrak{z}])$ is directly from d is metric.

Theorem 3.10
$(\hat{Y},\hat{d})$ and (X,d) are nearly isometry.
Proof
We have to find a bijective function between X *and* $\hat{Y}$ such that
$\hat{d}(f(x),f(y)) \simeq d(x,y)$,
Where $\hat{Y} = \{[S_x], x \in X\}$, $S_x = \{ s_n \in C: s_n \simeq x\ \forall^{unlimited} n\}$ and

$$\hat{d}([\lambda],[\mathfrak{z}]) = \begin{cases} d(\alpha_n,\beta_n) = \begin{cases} c & n\ unlimited\ \ \alpha_n \not\simeq \beta_n \\ \varepsilon & n\ unlimited\ \ \alpha_n \simeq \beta_n \end{cases} \\ 0 \qquad\qquad otherwise \end{cases}$$

Let $f(x) = [S_x]\ \forall x \in X$.
Now, we prove that f is one to one and onto
Let $.f(x) = f(y)$ We have to show that $[x]=[y]$,
Now, since $f(x)=f(y)$, then $[S_x]=[S_y]$, where
$[S_x] = \{ \alpha_n \in C: \alpha_n \simeq x\ \forall^{unlimited}\ n\}$
$[S_y] = \{ \beta_n \in C: \beta_n \simeq y\ \forall^{unlimited}\ n\}$
Since $[S_x]=[S_y]$, then $[S_x] \subseteq [S_y]$ and $[S_y] \subseteq [S_x]$, In other way, we can say for all unlimited n, if $\alpha_n \in S_y$, then $\alpha_n \simeq y$ and if $\beta_n \in S_x$, then $\beta_n \simeq x\ \forall^{unlimited}\ n$,
Therefore,
$\alpha_n \simeq x \simeq \beta_n \simeq y\ \forall^{unlimited}\ n$.
That is $x \simeq y$.
In general, if $\alpha_n \simeq \beta_n$, then $[x]=[y]$.
Hence, we prove that f is one to one.
To prove that f is onto
Let $y \in \hat{Y}$. Then there exist $x \in X$ such that $y = [S_x]$
That is $y = f(x)$
Hence, f is onto.
Now, we have to show that

$$\hat{d}(f(x),f(y)) \simeq d(x,y)$$

Let $x,y \in X$ such that $f(x) = [S_x]$ and $f(y) = [S_y]$. Then,

$$\hat{d}(f(x),f(y)) = \hat{d}([S_x],[S_y]) = \begin{cases} d(\alpha_n,\beta_n) = \begin{cases} c & n\ unlimited\ \ \alpha_n \not\simeq \beta_n \\ \varepsilon & n\ unlimited\ \ \alpha_n \simeq \beta_n \end{cases} \\ 0 \qquad\qquad otherwise \end{cases}$$

Then either $\hat{d}(f(x),f(y)) = d(\alpha_n,\beta_n)$ where $\alpha_n \to x$ and $\beta_n \to y\ \forall^{unlimited}\ n$.

That is $\hat{d}(f(x),f(y)) = d(\alpha_n,\beta_n) \simeq d(x,y)\ \forall^{unlimited}\ n$.
Or,
$\hat{d}(f(x),f(y)) = d(\alpha_n,\beta_n) = \varepsilon \simeq 0$ where $\alpha_n \to x$ and $\beta_n \to y\ \forall^{unlimited}\ n$.
Then
$\hat{d}(f(x),f(y)) = d(\alpha_n,\beta_n) \simeq d(x,y)\ \forall^{unlimited}\ n$.
In both cases we have

$$\hat{d}(f(x),f(y)) \simeq d(x,y)$$

Thus, $(\hat{Y},\hat{d})$ is nearly isometry (X,d).

Theorem 3.11
$(\hat{Y},d)$ is dense subspace of $(\hat{X},\hat{d})$.
Proof
To prove that $(\hat{Y},d)$ is dense subspace of $(\hat{X},\hat{d})$. we have to show that for all $p \in \hat{X}-\hat{Y}$, then p is a limit point of $\hat{Y}$. That is to show that for all $p \in \hat{X}-\hat{Y}$, there exists $q \in \hat{Y}$ such that $q \in mond(p)$, or we have to show that we can embedded $\hat{Y}$ into $\hat{X}$ isometrically.
Let $\hat{x} \in \hat{X}-\hat{Y}$. Then from **Definition 3.6**, we have that $\hat{x}$ is a Cauchy sequence in $\hat{X}$.
Let $\hat{x} \equiv x_n$ such that $x_n \to x$. Then by **Definition 3.6** we have either $x \in X$ *or* $x \notin X$.
Now, since x_n is Cauchy sequence, then $x_n \simeq x_m$ $\forall^{unlimited}\ n,m$.
Let $\{ t_n\}$ be a sequence in X such that $t_n \simeq x_k$ for all fixed k, that is $\{ t_n\}$ is a constant sequence. Then, its clear that $t_n \to x_k$. That is $t_n \simeq x_k\ \forall^{unlimited}\ n$.
From definition of S_x *and* $\hat{Y}$, we have $t_n \in \hat{Y}$ and for all unlimited n,
$\hat{d}([x],[x_k]) \equiv d(x_n,t_n) = \varepsilon \simeq 0\ \forall^{unlimited}\ n$ (because $x_n \simeq t_n$).
Therefore, $[x] \simeq [x_k]$. It means that $x \simeq x_k$.
Thus, $x_k \in mond(x)$.
That is $[x_k] \in [x] = \{x_n: x_n \to x\}$

Hence, $(\hat{Y}, d)$ is dense subspace of $(\hat{X}, \hat{d})$.
As a consequence of the all the previous given results in the following theorem, we will give prove of main considered problem of completion.

Theorem 3.12 (Basic Theorem)

$(\hat{X}, \hat{d})$ is complete metric space.

Proof

Since $\hat{X} = \{ [\lambda]: x_n \to \lambda \ x_n \in X\}$, where

$$[\lambda] = \begin{cases} x_n \in C : \ x_n \to \lambda \in X \\ y_n \in C : \ y_n \to \lambda \notin X \end{cases} \quad \text{and}$$

$$\hat{d}([\lambda], [\mathfrak{z}]) = \begin{cases} d(x_n, y_n) & if\ n\ is\ unlimited \\ 0 & otherwise \end{cases}$$

for $x_n, y_n \in C \subset X$ such that $x_n \to \lambda, y_n \to \mathfrak{z}$, and
$[\lambda] = [\mathfrak{z}] \equiv \lambda \simeq \mathfrak{z}$.
$\equiv x_n \simeq y_n \ \forall^{unlimited}\ n.$
Also $\hat{Y} = \{[S_x], x \in X\}\ and\ \ S_x = \{ s_n \in C: s_n \simeq x\ \forall^{unlimited}\ n\}$.
We will try to follow through the prove step by step:

Step1: Let $\{ [S_{x_n}]\}$ be a Cauchy sequence in $\hat{Y}$ where $[S_{x_n}]$ is equivalence class of all Cauchy sequences. Then the metric distance between $[S_{x_m}]$ and $[S_{x_m}]$ is equivalence to

$$\hat{d}([S_{x_m}], [S_{x_m}]) \equiv \begin{cases} d(S_{x_m}, S_{x_n}) & if\ n, m\ are\ unlimited \\ 0 & otherwise \end{cases}$$
$$\equiv \begin{cases} d(x_n, x_m) & if\ n, m\ are\ unlimited \\ 0 & otherwise \end{cases}$$

Since $\{ [S_{x_n}]\}$ is a sequence of equivalent classes of Cauchy sequences in $\hat{Y}$ and $\{ [S_{x_n}]\}$ is Cauchy sequence, it follows that $\{x_n\}$ is a Cauchy sequence in X.
Hence $x_n \simeq x_m\ \forall^{unlimited}\ n, m.$

Step2: Our aim in this step is to take an element in $\hat{X}$ and show that our Cauchy sequence $\{ [S_{x_n}]\}$ in $\hat{Y}$ is converge to an element in $\hat{X}$.
Now, let $[x] \in \hat{X}$. Then
$[x] = \{x_n \in C : \ x_n \simeq x \in X\ \forall^{unlimited}\ n \}$, and

$$\hat{d}([x], [y]) = \begin{cases} d(x_n, y_n) & if\ n\ is\ unlimited \\ 0 & otherwise \end{cases}$$
$$\equiv \begin{cases} d(x_n, y_n) = \begin{cases} c & n\ unlimited\ \ x_n \not\simeq y_n \\ \varepsilon & n\ unlimited\ \ x_n \simeq y_n \end{cases} \\ 0 \quad\quad otherwise \end{cases} \tag{7}$$

Without loss of generality, we can take $[y] = [S_{x_n}]$, where $[S_{x_n}] \in \hat{Y}$.
Since $\{x_n\}$ is a Cauchy sequence in X we can write $[x_m]$ as:
$[x_m] = \{\{x_k\}: x_k \simeq x_m\}$, where $m = 1, 2, 3, \ldots$.
Now, since $[\lambda] = [\mathfrak{z}] \equiv \lambda \simeq \mathfrak{z} \equiv \alpha_n \simeq \beta_n\ \forall^{unlimited}\ n$, then

$$\hat{d}([x], [S_{x_n}]) = \begin{cases} d(x_n, x_{kn}) = \begin{cases} c & n\ unlimited\ \ x_n \not\simeq x_{kn} \\ \varepsilon & n\ unlimited\ \ x_n \simeq x_{kn} \end{cases} \\ 0 \quad\quad otherwise \end{cases}$$

Since we have $x_n \simeq x_m \forall^{unlimited} n, m$ and $x_{kn} \simeq x_k$, then

$$\hat{d}([x], [S_{x_n}]) \equiv \hat{d}([x_m], [S_{x_{nm}}])$$
$$\equiv \begin{cases} d(x_m, x_{nm}) & m\ is\ unlimited\ and\ n\ fixed \\ 0 & otherwise \end{cases}$$
$$\equiv \begin{cases} d(x_m, x_n) & m\ is\ unlimited\ and\ n = 1,2,3, \ldots \\ 0 & otherwise \end{cases}$$

But we know that $x_n \simeq x_m$. Then $d(x_m, x_n) \simeq 0$.
Hence, a Cauchy sequence $\{ [S_{x_n}]\}$ in $\hat{Y}$ is converge to a point $[x] \in \hat{X}$.

Step3: the last step is to take a sequence in $\hat{X}$ and to show that it is converge to that point that belongs to $\hat{X}$.
Assume that $\{\hat{x}_n\}$ is a Cauchy sequence in $\hat{X}$, then

$$\hat{d}(\hat{x}_n, \hat{x}_m) = \begin{cases} d(x_{kn}, x_{km}) = \begin{cases} c & n \text{ unlimited } x_{kn} \not\simeq x_{km} \\ \varepsilon & n \text{ unlimited } x_{kn} \simeq x_{km} \end{cases}, \\ 0 \quad \text{otherwise} \end{cases}$$

and since $[x] \in \hat{X}$, then we have $\forall^{unlimited}\, n\ \exists \lambda \in X$ such that $\hat{x}_n = [\lambda]$. Therefore,

$$\hat{d}([\lambda], [\mathfrak{z}]) \equiv \hat{d}(\hat{x}_n, \hat{x}_m) \equiv \begin{cases} d(x_{kn}, x_{km}) = \begin{cases} c & n \text{ unlimited } x_{kn} \not\simeq x_{km} \\ \varepsilon & n \text{ unlimited } x_{kn} \simeq x_{km} \end{cases} \\ 0 \quad \text{otherwise} \end{cases}$$

$$\equiv \begin{cases} d(x_n, x_m) & \forall^{unlimited}\, n, m \\ 0 & \text{otherwise} \end{cases},$$

where $\hat{x}_m = [\mathfrak{z}]$ and $x_n \in [\lambda] = \{\hat{x}_k\}$ for fixed k.

Since $\{x_n\}$ is a Cauchy sequence in X, then $x_n \simeq x_m \forall^{unlimited}\, n, m.$ (8)

Therefore $\hat{d}(\hat{x}_n, \hat{x}_m) \equiv d(x_n, x_m) \simeq 0.$

Now, by **Theorem 3.11**, we have $(\hat{Y}, d)$ is dense subspace of $(\hat{X}, \hat{d})$.

Then $\forall^{unlimited}\, n$, there exist $[S_{y_n}] \in \hat{Y}$ such that $y_n \in [S_{y_n}] \in \hat{Y}$.

By (1) we have $x_n \simeq y_n \ \forall^{unlimited}\, n.$
(9)

Since $x_n \simeq y_n$ and $\{x_n\}$ is a Cauchy sequence in X, then $\{y_n\}$ is a Cauchy sequence in X.

Thus $x_m \simeq x_n \simeq y_n \forall^{unlimited} n, m$, in the other hand we have $x_m \simeq y_m$.

Then $y_n \simeq y_m \forall^{unlimited}\, n, m.$

Similarly, $\hat{d}(\hat{y}_n, \hat{y}_m) \equiv d(y_n, y_m) \simeq 0.$

Thus $\hat{y}_n \simeq \hat{y}_m \ \forall^{unlimited}\, n, m.$

Thus $\{\hat{y}_n\} \equiv \{[S_{y_n}]\}$ is a Cauchy sequence in $\hat{Y}$.

Again by **Theorem 3.11**, we have $\hat{Y}$ is nearly dense in $\hat{X}$.

Then there exist $\hat{y} \in \hat{X}$ such that $\hat{y}_n \simeq \hat{y} \ \forall^{unlimited}\, n.$ (10)

To complete the prove we have to prove that

$\hat{x}_n \simeq \hat{y} \ \forall^{unlimited}\, n.$ (11)

Now, by (7) and (8) we have $y_m \simeq x_n \ \forall^{unlimited} n, m.$

Returning the result to the correspond equivalent classes with respect to the metric $\hat{d}$, we obtain that

$$\hat{y}_m \simeq \hat{x}_n \ \ \forall^{unlimited}\, n, m.$$

By (10) we have $\hat{y}_m \simeq \hat{y} \ \forall^{unlimited}\, m.$

Hence, $\hat{x}_n \simeq \hat{y} \ \forall^{unlimited}\, n.$

That is a Cauchy sequence $\{\hat{x}_n\}$ in $\hat{X}$ is converges to $\hat{y}$ in $\hat{X}$. Thus $(\hat{X}, \hat{d})$ is complete metric space.

Theorem 3.13

If X is nearly complete, then $\hat{X}$ is completion of X, and either

$^{\circ}\varphi(X) = \hat{X}$ or $\varphi(X) \subseteq \hat{X}$.

Proof

Let $T = [\{t_n\}] \in \hat{X}$. Then $\{\varphi(t_k)\}_{k\in\mathbb{N}}$ is a sequence in $\varphi(X)$. We will try to prove that the sequence $\{\varphi(t_k)\}_{k\in\mathbb{N}} \simeq T \ \in \hat{X}$, that is to show that $\varphi(X)$ is nearly dense in $\hat{X}$.

Now,

$$\hat{d}(T, \varphi(t_k)) = \hat{d}(t_n, (\varphi(t_k))_n)$$

$$\equiv \begin{cases} d(t_n, \varphi(t_k)) & n \text{ unlimited} \\ 0 & \text{otherwise} \end{cases}$$

$$\equiv \begin{cases} d(t_n, t_k) & n, k \text{ unlimited} \\ 0 & \text{otherwise} \end{cases}$$

Since $\{t_n\}$ is Cauchy, then $t_n \simeq t_k$ $\forall^{unlimited}\, n, k$, and

$$\equiv \begin{cases} d(t_n, t_k) = \varepsilon & n, k \text{ unlimited} \\ 0 & \text{otherwise} \end{cases}$$

It means that $d(t_n, t_k) \simeq 0 \ \forall^{unlimited}\, n, k.$

Which means that $(t_k) \simeq T \ \ \forall^{unlimited}\, k.$

Then $\varphi(t_k) \to T \in \hat{X}$.

On the other hand, we have $\varphi(t_k) \in \varphi(X) \forall^{unlimited}\, k$

Then $\varphi(X)$ is nearly dense in $\hat{X}$. By **Lemma 3.5** we have either $^{\circ}\varphi(X) = \hat{X}$ or $\varphi(X) \subseteq \hat{X}$.

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