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RESEARCH PAPER

Darboux and rational first integrals for a family of cubic three dimensional system

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A B S T R A C T:
In this paper, we investigate the first integrals of the following system

$$\dot{x} = y, \quad \dot{y} = z, \qquad \dot{z} = b_1x + b_2yx^2 + b_3z^3 \tag{1}$$

where $b_1 \in \mathbb{R}$ and $b_2,\ b_3 \in \mathbb{R}/\{0\}$. This kind of system is a special case of three-dimensional polynomial cubic differential systems. Generally, several methods can be used to investigate the first integrals, but unfortunately, most of them are not enabled for finding first integrals. In this study, the Darboux method has been used to study the first integrals for the generalized system for all parameters. We characterize all its invariant algebraic surfaces and all its exponential factors of that system. We have shown that the above system does not admit a polynomial, rational, and Darboux first integrals for any values of the parameters.

KEY WORDS: Draboux First Integral; Invariant Algebraic Surfaces; Exponential Factor; Rational First integral.

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1.INTRODUCTION

Investigating the first integrals is one of the most famous problems in qualitative theory for three-dimensional polynomial differential systems, because the knowledge of first integrals of a differential system can be very useful in order to understand and simplify the study of the dynamics of the system. In 1878, Darboux (Darboux, G., 1878a, Darboux, G., 1878b) presented a new simple method to construct first integrals and integrating factors for the planar polynomial vector field using their invariant algebraic curves. The author showed how the first integrals of polynomial vector fields in $\mathbb{R}^2$ or $\mathbb{C}^2$ possessing sufficient invariant algebraic curves which can be constructed.

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In 1979 Jouanolou expanded the planar polynomial differential system into higher-dimensional, especially for rational first integrals. Existencing first integrals of Draboux type based on the existence of invariant algebraic surfaces and exponential factors. This kind of integrability has been successfully applied to study some physical models (see for instance(Jalal, A.A. et al., 2020, Barreira, L., et al., 2020, Llibre, J. and Zhang, X., 2009b, Llibre, J. and Valls, C., 2005a Llibre, J. and Valls, C., 2010c Llibre, J. and Valls, C., 2005a, Llibre, J. and Valls, C., 2007b, Valls, C., 2005)), limit cycles, centers and bifurcation problems of polynomial differential systems (see for instance (Giné, J. and Llibre, J., 2005, Llibre, J. and Rodrıguez, G., 2004, Schlomiuk, D., 1993)). The key existing and non existing of Darboux first integrals depend on exponential factors and invariant algebraic surfaces. More information about this method can be found in (Llibre, J. and Zhang, X., 2009a, Llibre, J. and Zhang, X., 2009a).

Now we apply the Darboux method for investigating the existence of analytic first integrals in one of the families of differential systems which for some values of the parameters exhibit chaos. More precisely, we study the Darboux first integral of the following three-dimensional differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \qquad (3) \\ \dot{z} &= b_1 x + b_2 y x^2 + b_3 z^3, \end{aligned}$$

where $b_1 \in \mathbb{R}$ and $b_2, b_3 \in \mathbb{R}/\{0\}$.
The above system has been reported in (Sprott, J.C., 1997), the author verifies the chaotic behaviours for system (3), where $0 < b_1 < 1$ and $b_2 = b_3 = -1$. The existence of sufficiently first integrals determines completely its phase portrait in a complete way for differential equations. Many different methods have been used for studying the existence of first integrals for polynomial systems. Generally, for a given differential system, it is a difficult problem to determine the existence or nonexistence of a first integral. For a given polynomial system. This paper aims to study the polynomial, rational, and the Darboux integrability of the system (3). This kind of integrability will be studied by using the Darboux theory of integrability, for more details see (Dumortier, F., et al. 2006) and (Llibre, J. and Zhang, X., 2009a, Llibre, J. and Valls, C., 2005a).

2. PRELIMINARIES

Firstly, we denote a vector field for the differential system (3) as follows

$$X = y\frac{\partial}{\partial x} + z\,\frac{\partial}{\partial y} + (b_1 x + b_2 y x^2 + b_3 z^3)\,\frac{\partial}{\partial z}.$$

Suppose that $\mathcal{W}$ be an open dense subset of $\mathbb{R}^3$. Let $\mathcal{H}: \mathcal{W} \to \mathbb{R}$ be $\mathcal{C}^1$. Then $\mathcal{H}$ is the first integral of vector field X in $\mathcal{W}$ if $\mathcal{H}$ is a constant on the solutions of system (3) contained in $\mathcal{W}$; i.e. if

$$X\mathcal{H} = y\frac{\partial \mathcal{H}}{\partial x} + z\,\frac{\partial \mathcal{H}}{\partial y} + (b_1 x + b_2 y x^2 + b_3 z^3)\frac{\partial \mathcal{H}}{\partial z} = 0, \qquad (4)$$

for all $(x, y, z) \in \mathcal{W}$.

We say that $h(x, y, z) = 0$ is an invariant algebraic surface (Darboux polynomial) of the system (3) where $h \in \mathbb{C}[x, y, z]\backslash\mathbb{C}$ if there exists $\mathcal{K} \in \mathbb{C}[x, y, z]$ such that

$$Xh = y\frac{\partial h}{\partial x} + z\,\frac{\partial h}{\partial y} + (b_1 x + b_2 y x^2 + b_3 z^3)\frac{\partial h}{\partial z} = \mathcal{K}h \qquad (5)$$

where $\mathcal{K}$ is called the cofactor of degree at most two.
A non-constant function $E = e^{g/h}$ is called an exponential factor of the polynomial system (3) if the following equation is satisfied

$$XE = y\frac{\partial E}{\partial x} + z\,\frac{\partial E}{\partial y} + (b_1 x + b_2 y x^2 + b_3 z^3)\frac{\partial E}{\partial z} = \mathcal{L}E \qquad (6)$$

where $g, h \in \mathbb{C}[x, y, z]$ are coprime polynomail and $\mathcal{L}$ is cofactor of the exponential factor with the degree of cofactors is almost two.

The next result explaining that how to find Darboux first integrals. The prove of the following theorem can be found in (Dumortier, F., et al. 2006).

Theorem 1. Assume that a polynomial vector field X of degree d in $\mathbb{C}^3$ admits p irreducible invariant algebraic surfaces $h_i = 0$; such that the h_i they are pairwise relatively coprimes with cofactors $\mathcal{K}_i$ for $i = 1, \dots, p$ and q exponential factors $E_i = e^{\frac{g_j}{h_j}}$ with cofactors $\mathcal{L}_j$ for j $1, \dots, q$.
There exist λ_i and μ_j not all zero such that

$$\sum_{i=1}^{p} \lambda_i \mathcal{K}_i + \sum_{j=1}^{q} \mu_j \mathcal{L}_j = 0 \qquad (7)$$

if and only if the (multi-valued) function

$$h_1^{\lambda_1} h_2^{\lambda_2} \dots h_p^{\lambda_p} \left(E_1^{\mu_1} E_2^{\mu_2} \dots E_q^{\mu_q}\right) \qquad (8)$$

is a first integral for X.

Darboux first integral is the first integral that can be obtained from equation (8). A polynomial first integral in the polynomial form. A rational function is called rational first integral if it is satisfied equation (4).

The following theorem has been proven in (Llibre, J. and Zhang, X., 2009a, Llibre, J. and Valls, C., 2005a) will be helpful to investigate the exponential factor of the system (3).

Theorem 2. The following statements hold.

a. If $e^{\frac{g}{h}}$ is an exponential factor for the polynomial differential system (1) and g is not a constant polynomial, then $h = 0$ is an invariant algebraic surface.
b. Ultimately e^{g} can be an exponential factor, derived from the multiplicity of the infinity invariant plane.

Theorem 3. Let X be a polynomial vector field defined in $\mathbb{C}^n$ of degree $d > 0$. Then X admits $\binom{d+n-1}{n} + n$ irreducible invariant algebraic hypersurfaces if and only if X has a rational first integral.

3. RESULTS AND THEIR PROOF

In this section, we will prove that the system has no Darboux first integral nither rational first integral. The existence and nonexistence of invariant algebraic surfaces play an important role in studying the Darboux type of the first integral for any differential system. Finally, we have found that the system (3) has no Draboux polynomial nither rational first integral.

Proposition 4. System (3) does not admit Draboux polynomials.

Proof. Let $h = \sum_{i=1}^{n} h_i\,(x,y,z)$ be a Darboux polynomial of the system (3), and $h_i(x,y,z)$ be a non zero homogeneous polynomial of degree i, for $i = 1, \ldots, n$. Since the system is cubic, then we can assume $\mathcal{K}$ of the following form $\mathcal{K} = k_0 + k_1x + k_2y + k_3z + k_4x^2 + k_5xy + k_6xz + k_7\,y^2 + k_8yz + k_9z^2$, for some $k_i \in \mathbb{R}$, and $j = 0, \ldots, 9$. Then, equation (5) must satisfy the following equation

$$y\frac{\partial h}{\partial x} + z\,\frac{\partial h}{\partial y} + \left(b_1x + b_2yx^2 + b_3z^3\right)\frac{\partial h}{\partial z} = (k_0 + k_1x + k_2y + k_3z + k_4x^2 + k_5xy + k_6xz + k_7\,y^2 + k_8yz + k_9z^2)h \qquad (9)$$

Counting the terms of degree $n+2$ in equation (9), we have

$$(b_2yx^2 + b_3z^3)\frac{\partial h_n}{\partial z} = (k_4x^2 + k_5xy + k_6xz\; k_7\,y^2 + k_8yz + k_9z^2)h_n.$$

Solving the above differential equation, we obtain

$$h_n = S_n(x,y)\,(b_2x^2y + b_3z^3)^{\frac{k_9}{3b_3}} \left(\frac{\sqrt{z^2 - z\sqrt[3]{\frac{b_2yx^2}{b_3}} + \left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}}{z + \sqrt[3]{\frac{b_2yx^2}{b_3}}} \right)^{A(x,y)} \qquad (10)$$

$$exp\left(\frac{1}{\sqrt{3}}arctan\left(\frac{2}{3}\frac{\sqrt{3}z}{\sqrt{\frac{b_2yx^2}{b_3}}} - \frac{\sqrt{3}}{3}\right)\frac{k_6x\sqrt[3]{\frac{b_2yx^2}{b_3}} + k_8y\sqrt[3]{\frac{b_2yx^2}{b_3}} + k_4x^2 + k_5xy + k_7\,y^2}{b_3\left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}\right)$$

where $S_n(x,y)$ is an arbitrary polynomial and

$$A(x,y) = \frac{k_6x\sqrt[3]{\frac{b_2yx^2}{b_3}} + k_8y\,\sqrt[3]{yx^2} - k_4x^2 - k_5xy - k_7\,y^2}{b_3\left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}.$$

Since h_n is a polynomial of degree n, it is required that $k_4 = k_5 = k_6 = k_7 = k_8 = 0$ and $k_9 = 3mb_3$, for some positive integer m. Now, we give terms of degree $n+1$ in equation (9) and we find that

$$(b_2yx^2 + b_3z^3)\frac{\partial h_{n-1}}{\partial z} = (k_1x + k_2y + k_3z\,)\,h_n + (3mb_3z^2)h_{n-1}.$$

Solving the above differential equation we obtain

$$h_{n-1} = \frac{P(x,y,z)T(x,y)}{\sqrt{3}}$$

$$arctan\left(\frac{1}{\sqrt[3]{\frac{b_2yx^2}{b_3}}}\left(-2z+\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)\right)$$

$$-3\mathrm{P}(x,y,z)\ \ b_3\ \ S_{n-1}(x,y)$$

$$\left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2 + \ \frac{P(x,y,z)V(x,y)}{2}$$

$$\ln\left(\frac{z^2 - z\sqrt[3]{\frac{b_2yx^2}{b_3}} + \left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}{\left(z+\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}\right)$$

where

$$P(x,y,z) = -\frac{(b_2yx^2 + b_3z^2)}{3b_3\left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}$$

$$T(x,y) = S(x,y)\left(k_1x+\ k_2y + k_3\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)$$

$$V(x,y) = S(x,y)\left(k_1x+\ k_2y - k_3\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)$$

and $S_{n-1}(x,y)$ is an arbitrary polynomial. Since h_{n-1} is a polynomial then either $S_n(x,y)=0$, or $k_1 = k_2 = k_3 = 0$. If $S_n(x,y)=0$, then we obtain $h_n = 0$, which is a contradiction, they must be $k_1 = \ k_2 = k_3 = 0$. Thus we have

h_{n-1}= $(b_2x^2y + b_3z^3)^m\ S_{n-1}(x,y)$.

Finally, calculating the terms of degree n in equation (9) gives the following equation

$$y\frac{\partial h_n}{\partial x} + z\ \frac{\partial h_n}{\partial y} + b_1x\ \frac{\partial h_n}{\partial z}\ + (b_2yx^2$$

$$+b_3z^3)\frac{\partial h_{n-2}}{\partial z} = k_0\ h_n + 3mz^2h_{n-2}.$$

Solving the above differential equation, we have

$$h_{n-2} = (3mb_1yx^2 - mxz^2 - 2my^2z)$$

$$S_n(x,y)A_1(x,y,z) - \frac{1}{3}x\sqrt[3]{\frac{b_2yx^2}{b_3}}A_2(x,y,z)$$

$$A_1(x,y,z)\left(S_nm + 3y\frac{\partial S_n}{\partial y}\right)(b_2x^2y + b_3z^3)$$

$$+\left(k_0xy - \frac{4m}{3}y^2 - xy^2\frac{\partial S_n}{\partial x}\right)A_1(x,y,z)$$

$$A_3(x,y,z)+(b_2x^2y + b_3z^3)^mS_{n-2}(x,y),$$

where $S_{n-2}(x,y)$ is an arbitrary polynomial and

$$A_1(x,y,z) = \frac{(b_2x^2y + b_3z^3)^{m-1}}{3\ b_3xy\left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2},$$

$$A_2(x,y,z) = \sqrt{3}\,\mathrm{arctan}\left(\frac{2}{\sqrt{3}}\frac{z}{\sqrt[3]{\frac{b_2yx^2}{b_3}}} - \frac{1}{\sqrt{3}}\right)$$

$$+ln\left(\frac{\sqrt{z^2 - z\sqrt[3]{\frac{b_2yx^2}{b_3}} + \left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}}{z+\sqrt[3]{\frac{b_2yx^2}{b_3}}}\right),$$

$$A_3(x,y,z) =$$

$$\sqrt{3}\,arctan\left(\frac{2}{\sqrt{3}}\frac{z}{\sqrt[3]{\frac{b_2yx^2}{b_3}}} - \frac{1}{\sqrt{3}}\right)$$

$$+\ ln\left(\frac{z+\sqrt[3]{\frac{b_2yx^2}{b_3}}}{\sqrt{z^2 - z\sqrt[3]{\frac{b_2yx^2}{b_3}} + \left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}}\right).$$

Since $h_{n-2}(x,y,z)$ is a homogeneous polynomial, then we must have

$$k_0xy - \frac{4m}{3}y^2 - xy^2\frac{\partial S_n}{\partial x} = 0, \qquad (11)$$

and

$$S_n m + 3y\frac{\partial S_n}{\partial y} = 0. \qquad (12)$$

From equation (11) , we obtain

$$S_n(x,y) = \frac{S(y)e^{\frac{k_0x}{y}}}{(\sqrt[3]{x})^{4m}} , \qquad (13)$$

where $S(y)$ is an arbitrary function. Now, by substituting equation (13) in equation (10) we obtain

$$h_n(x,y) = \frac{S(y)(b_2x^2y + b_3z^3)^m e^{\frac{k_0x}{y}}}{(\sqrt[3]{x})^{4m}}$$

Since h_n is homogeneous polynomial, then we have $k_0 = m = 0$, then $k_9 = 0$. Thus $k_j = 0$, for $j = 0,1,\dots,9$. This gives that the system has no Darboux polynomial.

Proposition 5. System (3) has no polynomial first integrals.

Proof. Let $\mathcal{H}(x,y,z)$ be a polynomial first integral of the degree n of the system (3). We can write $\mathcal{H} = \sum_{i=1}^{n}\mathcal{H}_i(x,y,z)$, where each $\mathcal{H}_i$ is a homogenous polynomial in the variables x, y, and z and $\mathcal{H}_i \neq 0$ for $n \geq 1$. Thus, $\mathcal{H}$ is satisfied equation (4). Then by calculating the terms of degree n + 2 in equation (4), we obtain

$$(b_2yx^2 + b_3z^3)\frac{\partial \mathcal{H}_n}{\partial z} = 0.$$

Since b_2, b_3 are not zero, then form above differential equation we obtain $\mathcal{H}_n = \mathcal{H}_n(x,y)$. The terms of degree $n+1$ in equation (4), we have

$$(b_2yx^2 + b_3z^3)\frac{\partial \mathcal{H}_{n-1}}{\partial z} = 0,$$

Since b_2, b_3 are not zero, then we obtain $\mathcal{H}_{n-1} = \mathcal{H}_{n-1}(x,y)$. Again, computing the terms of the degree n in equation (4), we have

$$y\frac{\partial \mathcal{H}_n}{\partial x} + z\frac{\partial \mathcal{H}_n}{\partial y} + (b_1x + b_2yx^2 + b_3z^3)\frac{\partial \mathcal{H}_{n-2}}{\partial z} = 0.$$

Solving the above differential equation, we obtain

$$\mathcal{H}_{n-2} = A_4(x,y,z)\left(\frac{y}{\sqrt[3]{\frac{b_2yx^2}{b_3}}}\frac{\partial \mathcal{H}_n}{\partial x} + \frac{\partial \mathcal{H}_n}{\partial y}\right) + A_5(x,y,z)\left(\frac{\partial \mathcal{H}_n}{\partial y} - \frac{y}{\sqrt[3]{\frac{b_2yx^2}{b_3}}}\frac{\partial \mathcal{H}_n}{\partial y}\right) + S_1(x,y).$$

where $S_1(x,y)$ is an arbitrary polynomial and

$$A_4(x,y,z) = \frac{-1}{\sqrt{3}b_3\sqrt[3]{\frac{b_2yx^2}{b_3}}} arctan\left(\frac{1}{\sqrt{3}}\left(\frac{2z}{\sqrt[3]{\frac{b_2yx^2}{b_3}}} - 1\right)\right),$$

$$A_5(x,y,z) = \ln\left(\frac{\sqrt[3]{z + \sqrt[3]{\frac{b_2yx^2}{b_3}}}}{\sqrt[6]{z^2 - z\sqrt[3]{\frac{b_2yx^2}{b_3}} + \left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}}\right)^{\frac{1}{b_3\sqrt[3]{\frac{b_2yx^2}{b_3}}}}.$$

Since $\mathcal{H}_{n-2}(x,y)$ is a homogeneous polynomial. Then it must be

$$\frac{\partial \mathcal{H}_n}{\partial y} - \frac{y}{\sqrt[3]{\frac{b_2yx^2}{b_3}}}\frac{\partial \mathcal{H}_n}{\partial y} = 0,$$

then $\mathcal{H}_n = S\left(ln\frac{\left(b_2^2x^5 + b_2x^2y\left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2\right)^{\frac{3}{5}}}{x^2}\right)$

This is a contradiction to our assumption that $\mathcal{H}_n$ be a homogeneous polynomial. This ends the proof of the proposition.

Proposition 6. The exponential factors of the system (3) are e^x, e^y, e^{x^2}, e^{y^2} and e^{xy} with cofactors $y, z, 2xy, 2yz,$ and $y^2 + xz$, respectively.

Proof. From Proposition (4) and Theorem (2), we can write exponential factors of the system (3) by $= e^{g(x,y,z)}$, where $g(x,y,z)$ is a polynomial of its variables. Since system (3) is cubic, we must have a cofactor of the form $\mathcal{L} = s_0 + s_1x + s_2y + s_3z + s_4x^2 + s_5xy + s_6xz + s_7y^2 + s_8yz + s_9z^2$ for some $s_j \in \mathbb{C}$ and $j = 0,1,\dots,9$. We write $g = \sum_{i=1}^{n} g_i(x,y,z)$, where each g_i is a homogeneous polynomial of degree i and assume that $g_n \neq 0$ for $n \geq 3$. Then E satisfy the following partial differential equation

$$y\frac{\partial e^g}{\partial x} + z\frac{\partial e^g}{\partial y} + (b_1x + b_2yx^2 + b_3z^3)\frac{\partial e^g}{\partial z} = \mathcal{L}e^g. \tag{14}$$

We simplify equation (14) to obtain

$$y\frac{\partial g}{\partial x} + z\frac{\partial g}{\partial y} + (b_1x + b_2yx^2 + b_3z^3)\frac{\partial g}{\partial z} = \mathcal{L}. \tag{15}$$

We compute the terms of degree n + 2 in equation (15) to obtain

$$(b_2yx^2 + b_3z^3)\frac{\partial g_n}{\partial z} = 0,$$

since $b_2, b_3 \in \mathbb{R}/\{0\}$, this implies that $g_n = g_n(x,y)$.

The terms of degree $n + 1$ in equation (15) give that

$$(b_2yx^2 + b_3z^3)\frac{\partial g_{n-1}}{\partial z} = 0,$$

since $b_2, b_3 \in \mathbb{R}/\{0\}$, then obtain $g_{n-1} = g_{n-1}(x,y)$.

Again we calculate the terms of the degree n in equation (15) to obtain

$$y\frac{\partial g_n}{\partial x} + z\frac{\partial g_n}{\partial y} + b_1x\frac{\partial g_n}{\partial z} + (b_2yx^2 + b_3z^3)\frac{\partial g_{n-2}}{\partial z} = 0.$$

Solving the above equation for g_{n-2}, we obtain

$$g_{n-2} = A_4(x,y,z)\left(\frac{y}{\sqrt[3]{\frac{b_2yx^2}{b_3}}}\frac{\partial g_n}{\partial x} + \frac{\partial g_n}{\partial y}\right) + A_5(x,y,z)\left(\frac{\partial g_n}{\partial y} - \frac{y}{\sqrt[3]{\frac{b_2yx^2}{b_3}}}\frac{\partial g_n}{\partial y}\right) + S_0(x,y),$$

where $S_0(x,y)$ is an arbitrary polynomial and

$$A_4(x,y,z) = \frac{-1}{\sqrt{3}b_3\sqrt[3]{\frac{b_2yx^2}{b_3}}}\, arctan\left(\frac{1}{\sqrt{3}}\left(\frac{2z}{\sqrt[3]{\frac{b_2yx^2}{b_3}}} - 1\right)\right),$$

$$A_5(x,y,z) = \ln\left(\frac{\sqrt[3]{z + \sqrt[3]{\frac{b_2yx^2}{b_3}}}}{\sqrt[6]{z^2 - z\sqrt[3]{\frac{b_2yx^2}{b_3}} + \left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2}}\right)^{\frac{1}{b_3\sqrt[3]{\frac{b_2yx^2}{b_3}}}}.$$

Since g_{n-2} is a homogeneous polynomial. Then it must be

$$\frac{\partial g_n}{\partial y} - \frac{y}{\sqrt[3]{\frac{b_2yx^2}{b_3}}}\frac{\partial g_n}{\partial y} = 0,$$

then $g_n = S\left(ln\,\frac{\left(b_2^2x^5 + b_2x^2y\left(\sqrt[3]{\frac{b_2yx^2}{b_3}}\right)^2\right)^{\frac{3}{5}}}{x^2}\right).$

Since g_n is a homogenous polynomial of degree n, then it must be $g_n = 0$, which is a contradiction. Then g must be a polynomial of the degree two satisfying equation (15). Suppose that

$$g(x,y,z) = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z + \beta_4 x^2 + \beta_5 xy + \beta_6 xz + \beta_7 y^2 + \beta_8 yz + \beta_9 z^2,$$

for some $\beta_k \in \mathbb{R}$, and $k = 0, \dots, 9$. Equation (14) yields

$$y\frac{\partial g}{\partial x} + z\frac{\partial g}{\partial y} + (b_1 x + b_2 y x^2 + b_3 z^3)\frac{\partial g}{\partial z} = s_0 + s_1 x + s_2 y + s_3 z + s_4 x^2 s_5 xy + s_6 xz + s_7 y^2 + s_8 yz + s_9 z^2,$$

solving the above equation when $b_2, b_3 \in \mathbb{R}/\{0\}$, then we have

Hence, $e^{s_2 x + s_3 y + \frac{1}{2} s_5 x^2 + \frac{1}{2} s_8 y^2 + s_7 xy}$ is the exponential factor with cofactor $s_2 y + s_3 z + s_5 xy + s_8 yz + s_7 y^2 + s_7 xz$. Then the result follows.

Theorem 7. System (3) has no Darboux first integrals.

Proof. Suppose that system (3) has a Darboux first integral. In viewing Proposition (4), system (3) does not admit invariant algebraic surface with non-zero cofactors and by Proposition (6) we have that system (3) has only five exponential factors e^x, e^y, e^{x^2}, e^{y^2} and e^{xy} with cofactors y, z, $2xy$, $2yz$, and $y^2 + xz$, respectively. From equation (7), there exists $\mu_i \in \mathbb{R}$ not all zero for $i = 1, \dots, 5$ such that

$$\mu_1 y + \mu_2 z + 2\mu_3 xy + 2\mu_4 yz + \mu_5 (y^2 + xz) = 0,$$

the above equation has only one solution which is $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$. This is a contradiction to the Darboux theorey. Hence system (3) has no Darboux first integrals.

Corollary 8. System (3) has no rational first integrals.

Proof. Suppose that system (3) has a rational first integral. Observably, the vector filed associated system (3) is defined in $\mathbb{R}^3$ with degree (d = 3). By utilizing Theorem (3), we must have $\binom{5}{3} + 3$, invariant algebraic surfaces. According to Proposition (4), system (3) has no invariant algebraic surfaces. This means that the system has no rational first integrals.

4. CONCLUSION

In this paper, we have applied a Darboux method for studying polynomial, rational, and Darboux first integrals of a three-dimensional jerk cubic system (3). According the definitions of invariant algebraic surfaces and exponential factors, we found all invariant algebraic surfaces and exponential factors of that system. We have shown that system (3) has no invariant algebraic surface. Moreover, this system has five exponential factors. Depending on the Darboux theory, we proved that the system has no polynomial, rational, and Draboux first integrals.

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