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RESEARCH PAPER

Bifurcation analysis for Shil'nikov Chaos Electro-dissolution of Copper.

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A B S T R A C T:
This paper is devoted to study the local bifurcations and stability of three dimensional systems that representing a Shil'nikov chaos during copper electro-dissolution. The local stability analysis of equilibrium points has been studied. It is shown that transcritical bifurcation can appears in the system. Also, the existence of Hopf bifurcation of the system around the equilibrium points is studied when the parameter passes through the critical value. Normal form theory is used to study bifurcating periodic solutions.

KEY WORDS: Local stability, Transcritical bifurcation, Hopf bifurcation, Copper electro-dissolution.
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1.INTRODUCTION :

For the qualitative investigation of dynamical systems bifurcation is the most vital theory, and it can be utilized to a expose complicated dynamical behaviors of the system under investigation. One of the more classical problems in the qualitative theory of polynomial differential systems in three dimension is to characterize the existence of periodic solutions. Hopf bifurcation provides the simplest criterion for a family of periodic solutions to bifurcate from known family of equilibrium points of dynamical system.

A Hopf bifurcation in $\mathbb{R}^3$ takes place in an equlilbrium point with eigenvalues of the form eigenvalues $\pm i\omega$, and the other eigenvalue $\lambda \neq 0$. Also, a Hopf point is said to be transversal if the parameter-dependent complex eigenvalue cross the imaginary axis with non-zero zero derivative.

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The specific part of local bifurcation is transcritical bifurcation. It is categorized by an equilibrium having an eigenvalue whose actual part passes through zero. In the transcritical bifurcation, equilibrium points are not destroyed neither created, but for a critical value of the parameter they switch stability. We study the local bifurcation theory of vector fields, for details about local bifurcation theory (see the books (Kuznetsov, 2013; Stephen, 2009)).

In 1988 a series of experimental investigations on the electrodissolution of a copper rotating disk in a $H2SO4/NaCl$ solution have been studied (Basset & Hudson, 1988). Based on their experimental time-series they presented evidence of Shilnikov chaos. According to the evidence, they reported a comparison of their time-series study with those obtained from a mathematical model considered previously by (Richetti, et al., 1987) in a discussion concerning homoclinic chaos in the Belousov–Zhabotinski reaction. The model is defined by the following three-dimensional ordinary differential equations

$$\dot{X} = Y,$$
$$\dot{Y} = Z, \quad (1)$$
$$\dot{Z} = -Z - 1.3Y - \mu X + X^2 - 1.425Y^2 + 0.2XZ - 0.01X^2Z.$$

where X, Y and Z represent chemical concentrations and $\dot{X}, \dot{Y}, \dot{Z}$ are their corresponding time derivatives and a is a real parameter. The qualitative behaviors such as homoclinic (Shil'nikov) and Hopf bifurcation of this system has been studied, see for instance (Basset & Hudson, 1988; Richetti, et al., 1987). The authors are only mentioned the Hopf bifurcation appear for this system without computations and analysis, but in this paper, we analysis Hopf bifurcation at all equilibrium points and investigate presents the formulae for determining the direction of the Hopf bifurcations and the stability of bifurcating periodic solutions by applying the normal form theory have been studied in (Hassard & Wan, 1978).

We have two key objectives in this study. The first objective is to demonstrate that system (1) undergoes a transcritical bifurcation, in which an exchange of stability occurs of system (1) that takes place at equilibrium points for certain bifurcation values of the parameters of the system. The transcritical bifurcation of system (1) by applying Sotomayor's Theorem via means of (Perko, 2006), happening in this system will be fulfilled with respect to the parameter a. The second objective is to investigate the existence of the Hopf bifurcation of system (1) by applying the normal form theory introduce by (Hassard & Wan, 1978). The Hopf bifurcation is also a kind of typical local bifurcation. It is the birth of a limit cycle (isolated closed orbits) from an equilibrium point of system. The study of Hopf bifurcation in three-dimensional dynamical system attracted the attention of many researchers (Junze, et al., 2018; Jiang , et al., 2010; Liu, et al., 2012). By selecting a suitable bifurcation parameter, There is good interest shown in finding of Hopf and transcritical bifurcations for some 3- dimensional quadratic polynomial differential systems; as an example (Toniol & Llibre, 2017; Sang & Huang, 2020).

The rest of this study is organized as follows: In section 2, we analyze transcritical bifurcation applying by Sotomayor's theorem, and we also, study the local stability of equilibrium points of system (1) by appling Routh Hurwitz criteria. In Section 3, we analyze Hopf bifurcation at equilibrium poins of system (1) as well as the stability of the bifurcation limit cycles via Hopf bifurcation theorem. Finally, conclusions are summarized in Section 4. (1)

2. Transcritical Bifurcations and Local Stability of Equilibrium Points of System (1)

In this section, we investigate trancritical bifurcations of system (1) by using Sotomayor's Theorem (Perko, 2006), and local stability of equilibrium points of system (1) via Routh Hurwitz criteria.

Theorem 2.1. For system (1), transcritical bifurcation occurs at the origin as $\boldsymbol{a}$ pass through zero.

Proof. We note that in system (1), a simple evaluation of transcritical bifurcation will be performed with regard to the parameter a. We have $a = 0$ and the vector field f associated with system (1) is given by

$$f(\chi, a) = (Y, Z, -Z - 1.3Y - aX + X^2 - 1.425Y^2 + 0.2XZ - 0.01X^2Z),$$

where $\chi = (X, Y, Z) \in \mathbb{R}^3$. The vector field f has only two equilibrium points origin $O(0,0,0)$ and $E(a, 0,0)$. If $a = 0$, then f has only one equilibrium point at the origin. Moreover, when $a = 0$, the Jacobian matrix of system (1) at origin is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{13}{10} & -1 \end{pmatrix},$$

the characteristic equation of J is

$$P(\lambda) = \lambda^3 + \lambda^2 + \frac{13}{10}\lambda = 0,$$

whose the roots are $\lambda_3 = 0, \lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{105}}{10}$. The transcritical bifurcation is characterized by the exchange of stability at the origin when parameter a passes through the bifurcation value $a = 0$. Thus, we will use Sotomayor's Theorem (Perko, 2006) to show that system (1) exhibits a transcritical bifurcation. Note that, the vectors $v = (1,0,0)$ and $w = (\frac{13}{10}, 1,1)$ are eigenvectors of the matrices J and

J^T, respectively, corresponding to the eigenvalue $\lambda_1 = 0$. Furthermore, we have that

$$w^T(\frac{\partial f}{\partial a})(0,0) = 0,$$

$$w^T(D_\chi\left(\frac{\partial f}{\partial a}\right)(0,0)v) = -1 \neq 0,$$

$$w^T(D_\chi{}^2 f(0,0)(v,v)) = 2 \neq 0.$$

Hance, all the hypotheses of Sotomayor's theorem are satisfied. Therefore, system (1) admits a transcritical bifurcation at the equilibrium point the origin at the bifurcation value $a = 0$. ■

Remark 2.2. System (1) neither admits saddle-node nor pitchfork bifurcations about equilibrium points because the second and fourth conditions in Sotomayor's Theorem (Perko, 2006), page [338–339**]** are not satisfied.

Proposition 2.3. For system (1)

i. If $0 < \mathrm{a} < \frac{13}{10}$, then the origin is asymptotically stable and if $\mathrm{a} < 0$ or $\mathrm{a} > \frac{13}{10}$, then the origin is unstable.

ii. If $-\frac{370}{13} + \frac{200}{13}\sqrt{3} < \mathrm{a} < 0$, or $\mathrm{a} < -\frac{370}{13} - \frac{200}{13}\sqrt{3}$, then equilibrium point $\mathrm{E(a,0,0)}$ is asymptotically stable. If $-\frac{370}{13} - \frac{200}{13}\sqrt{3} < \mathrm{a} < -\frac{370}{13} + \frac{200}{13}\sqrt{3}$ or $\mathrm{a} > 0$, then equilibrium point $\mathrm{E(a,0,0)}$ is unstable.

Proof.

i.The Jacobian matrix of system (1) at $\mathrm{O}(0,0,0)$ is

$$\mathrm{J_0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mathrm{a} & -\dfrac{13}{10} & -1 \end{pmatrix},$$

the characteristic equation of $\mathrm{J_0}$ is

$$\lambda^3 + \lambda^2 + \frac{13}{10}\lambda + a = 0. \qquad (3)$$

By the hypothesis and Routh-Hurwitz criterion (Llibre & Valls, 2011), then zeros of the equation (3) have negative real parts, hence the origin is asymptotically stable. But if $a < 0$ or $a > \frac{13}{10}$, then at least one zero of the equation (3) have positive real part, then the origin is unstable.

ii.The Jacobian matrix of system (1) at $E(a,0,0)$ is

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -\dfrac{13}{10} & -\dfrac{(a-10)^2}{100} \end{pmatrix},$$

the characteristic equation of J_1 is

$$\lambda^3 + \frac{(a-10)^2}{100}\lambda^2 + \frac{13}{10}\lambda - a = 0. \qquad (4)$$

Also, by means of the Routh-Hurwitz criterion, for equation (4) the real parts of the roots are all negative if and only if $a < 0$ and $\frac{13\,(a-10)^2}{1000} + a > 0$. Then must be $-\frac{370}{13} + \frac{200}{13}\sqrt{3} < a < 0$ or $a < -\frac{370}{13} - \frac{200}{13}\sqrt{3}$, therefore the equilibrium point $E(a,0,0)$ is asymptotically stable. On other hand, if $-\frac{370}{13} - \frac{200}{13}\sqrt{3} < \mathrm{a} < -\frac{370}{13} + \frac{200}{13}\sqrt{3}$ or $a > 0$, then equilibrium point $E(a,0,0)$ is unstable. ■

3. Hopf Bifurcation Analysis of System (1)

We investigate direction and stability of a periodic orbits of system (1) under some conditions. Moreover, we analyze Hopf bifurcations at equilibria $O(0,0,0)$ and $E(a,0,0)$ with $a \neq 0$ and obtain expressions and stabilities of the corresponding bifurcating periodic orbits.

3.1. Hopf Bifurcation Analysis at the Origin:

Proposition 3.1. (Existence of Hopf bifurcation). The characteristic equation (3) has negative eigenvalue $\lambda_3 = -1$ and pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\,\omega = \pm i\frac{\sqrt{130}}{10}$ if and only if $a = \frac{13}{10}$, then system (1) undergoes a Hopf bifurcation at the origin .

Proof. It is easy to show that equation (3) has negative eigenvalue and pair of purely imaginary eigenvalues if and only if $a = \frac{13}{10}$. Then if $a = \frac{13}{10}$ equation (3) becomes

$$\lambda^3 + \lambda^2 + \frac{13}{10}\lambda + \frac{13}{10} = 0,$$

hence

$$(\lambda + 1)(10\lambda^2 + 13) = 0. \qquad (5)$$

Therefore, equation (5) has one real root $\lambda_3 = -1$ and two purely imaginary eigenvalues $\lambda_{1,2} = \pm i\frac{\sqrt{130}}{10}$. We can choose a as the bifurcation parameter, and the critical value is $a = a_0 = \frac{13}{10}$. From to equation (3),we have

$$\frac{d\lambda}{da} = -\frac{10}{30\,\lambda^2 + 20\,\lambda + 13}.$$

Using the implicit function theorem, then transversality condition satisfies

$$\mathrm{Re}(\frac{d\lambda}{da})|_{a=a_0,\lambda=i\frac{\sqrt{130}}{10}} = \frac{5}{23} \neq 0,$$

and

$$\mathrm{Im}(\frac{d\lambda}{da})|_{a=a_0,\lambda=i\frac{\sqrt{130}}{10}} = \frac{5\sqrt{130}}{299}.$$

Hence, first and second conditions of Hopf bifurcation are satisfied. Nevertheless, for applying the Hopf bifurcation Theorem (Guckenheimer & Holmes, 2013) and system (1) undergoes a Hopf bifurcation at the origin. ■

We now, can study the stability, direction and period of periodic solution of system (1) at the origin.

Theorem 3.2. The Hopf bifurcation of system (1) at the origin is non-degenrate and subcritical. The period of periodic solution and its characteristic exponent are $\beta_2 = \frac{1258657}{14830400}, \tau_2 = -\frac{695827}{5029440}$ respectively.

Proof. Using the eigenvectors of the Jacobian matrix J_0 as the basis for a new coordinate system equation. By simple calculation, the eigenvectors corresponding to the eigenvalues $\lambda_1 = i\frac{\sqrt{130}}{10}$ and $\lambda_3 = -1$ are $v_1 + iv_2$ and v_3 respectively, where

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{13}{10} \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ \frac{\sqrt{130}}{10} \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Define

$$M = (v_2, v_1, v_3) = \begin{pmatrix} 0 & 1 & 1 \\ \frac{\sqrt{130}}{10} & 0 & -1 \\ 0 & -\frac{13}{10} & 1 \end{pmatrix}.$$

For system (1), the transformation is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

gives

$$\begin{aligned} \dot{u} &= -\frac{\sqrt{130}v}{10} + P(u,v,w), \\ \dot{v} &= \frac{\sqrt{130}u}{10} + Q(u,v,w), \qquad (6) \\ \dot{w} &= -w + R(u,v,w), \end{aligned}$$

where

$$\begin{aligned} P(u,v,w) &= \frac{\sqrt{130}(13\,v-10\,w)(v+w)^2}{29900} + \frac{57\,uw}{46} + \frac{37\sqrt{130}v^2}{1495} + \frac{97\sqrt{130}vw}{1495} - \frac{9\sqrt{130}w^2}{1196} - \frac{57\sqrt{130}u^2}{920}, \\ Q(u,v,w) &= -\frac{(13\,v-10\,w)(v+w)^2}{2300} - \frac{37\,v^2}{115} - \frac{97\,vw}{115} + \frac{9\,w^2}{92} + \frac{741\,u^2}{920} - \frac{57\sqrt{130}uw}{460}, \\ R(u,v,w) &= \frac{(13\,v-10\,w)(v+w)^2}{2300} + \frac{37\,v^2}{115} + \frac{97\,vw}{115} - \frac{9\,w^2}{92} - \frac{741\,u^2}{920} + \frac{57\sqrt{130}uw}{460}. \end{aligned}$$

Accorcding to the method in (Hassard & Wan, 1978) for system (6) we obtain

$$g_{11} = \frac{1}{4}\left(\frac{\partial^2 P}{\partial u^2} + \frac{\partial^2 P}{\partial v^2} + i\left(\frac{\partial^2 Q}{\partial u^2} + \frac{\partial^2 Q}{\partial v^2}\right)\right) = \frac{89\, i}{368} - \frac{89\sqrt{130}}{4784},$$

$$g_{02} = \frac{1}{4}\left(\frac{\partial^2 P}{\partial u^2} - \frac{\partial^2 P}{\partial v^2} - 2\frac{\partial^2 Q}{\partial u \partial v} + i\left(\frac{\partial^2 Q}{\partial u^2} - \frac{\partial^2 Q}{\partial v^2} + 2\frac{\partial^2 P}{\partial u \partial v}\right)\right) = \frac{1037\, i}{1840} - \frac{1037\sqrt{130}}{23920},$$

$$g_{20} = \frac{1}{4}\left(\frac{\partial^2 P}{\partial u^2} - \frac{\partial^2 P}{\partial v^2} + 2\frac{\partial^2 Q}{\partial u \partial v} + i\left(\frac{\partial^2 Q}{\partial u^2} - \frac{\partial^2 Q}{\partial v^2} - 2\frac{\partial^2 P}{\partial u \partial v}\right)\right) = \frac{1037\, i}{1840} - \frac{1037\sqrt{130}}{23920},$$

$$G_{21} = \frac{1}{8}\left(\frac{\partial^3 P}{\partial u^3} + \frac{\partial^3 P}{\partial u \partial v^2} + \frac{\partial^3 Q}{\partial u^2 \partial v} + \frac{\partial^3 Q}{\partial v^3} + i\left(\frac{\partial^3 Q}{\partial u^3} + \frac{\partial^3 Q}{\partial u \partial v^2} - \frac{\partial^3 P}{\partial u^2 \partial v} - \frac{\partial^3 P}{\partial v^3}\right)\right) = -\frac{39}{9200} - \frac{3\, i}{9200}\sqrt{130}. \quad (7)$$

We also have to find the solution of

$$h_{11} = \frac{1}{4}\left(\frac{\partial^2 R}{\partial u^2} + \frac{\partial^2 R}{\partial v^2}\right) = -\frac{89}{368},$$
$$h_{20} = \frac{1}{4}\left(\frac{\partial^2 R}{\partial u^2} - \frac{\partial^2 R}{\partial v^2} - 2\, i\frac{\partial^2 R}{\partial u \partial v}\right) = -\frac{1037}{1840}. \quad (8)$$

The solution of

$$\lambda_3 \phi_{11} = -h_{11},$$
$$(\lambda_3 - 2\, i\omega)\phi_{20} = -h_{20}. \quad (9)$$

is

$$\phi_{11} = -\frac{89}{368},$$
$$\phi_{20} = -\frac{1037}{11408} + \frac{1037\, i}{57040}\sqrt{130}. \quad (10)$$

Furthermore, we have

$$G_{110} = \frac{1}{2}\left(\frac{\partial^2 P}{\partial u \partial w} + \frac{\partial^2 Q}{\partial v \partial w} + i\left(\frac{\partial^2 Q}{\partial u \partial w} - \frac{\partial^2 P}{\partial v \partial w}\right)\right) = \frac{91}{460} - i\frac{1129}{11960}\sqrt{130},$$

$$G_{101} = \frac{1}{2}\left(\frac{\partial^2 P}{\partial u \partial w} - \frac{\partial^2 Q}{\partial v \partial w} + i\left(\frac{\partial^2 Q}{\partial u \partial w} + \frac{\partial^2 P}{\partial v \partial w}\right)\right) = \frac{479}{460} - i\frac{353}{11960}\sqrt{130},$$

$$g_{21} = G_{101}\phi_{20} + 2\, G_{110}\phi_{11} + G_{21} = -\frac{818807}{6559600} + \mathrm{i}\frac{45671757}{682198400}\sqrt{130}. \quad (11)$$

The critical values of Hopf bifurcation can be computed as follows

$$C_1(0) = \frac{i}{2\omega}\left(g_{20}g_{11} - 2\,(|g_{11}|)^2 - \frac{(|g_{02}|)^2}{3}\right) + \frac{1}{2}g_{21} = \frac{1258657}{29660800} + \mathrm{i}\frac{2472499}{144596400}\sqrt{130},$$
$$\mu_2 = -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(a_0))} = -\frac{1258657}{6448000},$$
$$\beta_2 = 2\,\mathrm{Re}(C_1(0)) = \frac{1258657}{14830400},$$
$$\tau_2 = -\frac{\mathrm{Im}(C_1(0)) + \mu_2 \mathrm{Im}(\lambda'(a_0))}{\omega} = -\frac{695827}{5029440}. \quad (12)$$

Since $\mu_2 < 0$, $\beta_2 > 0$ and $\tau_2 < 0$, then Hopf bifurcation is subcritical and non-degenrate with periodic orbit and it is unstable. Also the period of bifurcating periodic solutions decreases, which indicates that there is an orbitally unstable limit cycles. ■

3.2. Hopf Bifurcation Analysis at $E(a,0,0)$*:*

Similarly we can study the Hopf bifurcation at the equilibrium point $E(a,0,0)$ with $a \neq 0$.

Proposition 3.3. (Existence of Hopf bifurcation). If $a = -\frac{370}{13} + \frac{200\sqrt{3}}{13}$ or $a = -\frac{370}{13} - \frac{200\sqrt{3}}{13}$, the characteristic equation (4) has three eigenvalues one is negative root, the other two a pair purely imaginary conjugate roots, then system (1) undergoes a Hopf bifurcation at the equilibrium point $E(a,0,0)$.

Proof. Suppose that the characteristic equation (4) have roots $\lambda_3 \in \mathbb{R}$ and $\lambda_{1,2} = \pm i\omega$, where $\omega \in \mathbb{R}$. Then we can obtain that

$$(\lambda - \lambda_3)(\lambda^2 + \omega^2) = \lambda^3 + \frac{(a-10)^2}{100}\lambda^2 + \frac{13}{10}\lambda - a,$$

and simple calculations shows

$$\lambda_3 = -\frac{(a-10)^2}{100}, \omega^2 = \frac{13}{10}, \lambda_3\omega^2 = a,$$

then, it is easy to obtain that equation (4) has two real roots $\lambda_3 = -\frac{3700}{169} + \frac{2000\sqrt{3}}{169}$ or $\lambda_3 = -\frac{3700}{169} - \frac{2000\sqrt{3}}{169}$, and two purely imaginary eigenvalues $\lambda_{1,2} = \pm i\frac{\sqrt{130}}{10}$, if and only if $a = a_1 = -\frac{370}{13} + \frac{200\sqrt{3}}{13}$ or $a = a_2 = -\frac{370}{13} - \frac{200\sqrt{3}}{13}$ respectively. According equation (4), we have

$$\frac{d\lambda}{da} = -\frac{a\lambda^2-10\lambda^2-50}{a^2\lambda+150\lambda^2-20a\lambda+100\lambda+65}.$$

Using the implicit function theorem, then transversality condition satisfies

$$\mathrm{Re}\left(\frac{d\lambda}{da}\right)\Big|_{a=a_1,\lambda=i\frac{\sqrt{130}}{10}} = -\frac{514542586\sqrt{3}}{16684226809} - \frac{888000000}{16684226809},$$

$$\mathrm{Im}\left(\frac{d\lambda}{da}\right)\Big|_{a=a_1,\lambda=i\frac{\sqrt{130}}{10}} = -\frac{756257800\sqrt{130}\sqrt{3}}{216894948517} - \frac{1173636000\sqrt{130}}{216894948517},$$

and

$$\mathrm{Re}\left(\frac{d\lambda}{da}\right)\Big|_{a=a_2,\lambda=i\frac{\sqrt{130}}{10}} = \frac{514542586\sqrt{3}}{16684226809} - \frac{888000000}{16684226809},$$

$$\mathrm{Im}\left(\frac{d\lambda}{da}\right)\Big|_{a=a_2,\lambda=i\frac{\sqrt{130}}{10}} = -\frac{1173636000\sqrt{130}}{216894948517} + \frac{756257800\sqrt{130}\sqrt{3}}{216894948517}.$$

Thus, the first and second conditions for Hopf bifurcation are fulfilled by applying the Hopf bifurcation Theorem (Guckenheimer & Holmes, 2013), and system (1) displayes a Hopf bifurcation at the equilibrium point E. ■

We also, study the stability, direction and period of periodic solution of system (1) at the equilibrium point $E(a,0,0)$ via the normal form theory (Hassard & Wan, 1978).

Theorem 3.4.

i. The Hopf bifurcation of system (1) at the equilibrium point $E(a,0,0)$, when $a = a_1 = -\frac{370}{13} + \frac{200\sqrt{3}}{13}$ is non-degenrate and subcritical. The period of periodic solution and its characteristc exponent are respectively

$$\beta_2 - \frac{4079891710775285736627}{199515313237517536901200} - \frac{11753472814476305975771\sqrt{3}}{997576566187587684506000},$$

$$\tau_2 = -\frac{15221689654211}{248733043661440} - \frac{218125707829\sqrt{3}}{6218326091536}.$$

ii. The Hopf bifurcation of system (1) at the equilibrium point $E(a,0,0)$, when $a = a_2 = -\frac{370}{13} - \frac{200\sqrt{3}}{13}$ is non-degenrate supercritical. The period of periodic solution and its characteristic exponent are respectively

$$\beta_2 - \frac{4079891710775285736627}{199515313237517536901200} + \frac{11753472814476305975771\sqrt{3}}{997576566187587684506000},$$

$$\tau_2 = -\frac{15221689654211}{248733043661440} + \frac{218125707829\sqrt{3}}{6218326091536}.$$

Proof. Since the arguments of the computation for the case i and ii are very similar, we only prove case i .

At the equilibrium point $E(a,0,0)$, the change of variables $(X_1,Y_1,Z_1) = (X-a,Y,Z)$ transform system (1) into

$$\begin{aligned}\dot{X}_1 &= Y_1,\\ \dot{Y}_1 &= Z_1, \qquad\qquad (13)\\ \dot{Z}_1 &= \frac{(a-10)^2}{100}Z_1 + aX_1 - \frac{13}{10}Y_1 + {X_1}^2 - \frac{57}{40}{Y_1}^2 + \frac{1}{5}Z_1X_1 - \frac{a}{50}X_1Z_1 - \frac{1}{100}{X_1}^2Z_1.\end{aligned}$$

Using the eigenvectors of Jacobian matrix J_1 as the basis for a new coordinate system at the $E(a,0,0)$, when $a = a_1 = -\frac{370}{13} + \frac{200\sqrt{3}}{13}$, the eigenvector $u_1 = (1, \frac{i}{10}\sqrt{130}, -\frac{13}{10})^T$, associated with $\lambda_1 = i\frac{\sqrt{130}}{10}$ and the eigenvector $u_3 = (1, -\frac{3700}{169} + \frac{2000\sqrt{3}}{169}, \frac{25690000}{28561} - \frac{14800000\sqrt{3}}{28561})^T$, associated with $\lambda_3 = -\frac{3700}{169} + \frac{2000\sqrt{3}}{169}$, define

$$M = (v_2, v_1, v_3) = \begin{pmatrix} 0 & 1 & 1\\ \frac{\sqrt{130}}{10} & 0 & -1\\ 0 & -\frac{13}{10} & 1\end{pmatrix}.$$

The transformation

$$\begin{pmatrix}X_1\\Y_1\\Z_1\end{pmatrix} = B\begin{pmatrix}X_2\\Y_2\\Z_2\end{pmatrix},$$

imply system (13) to

$$\begin{aligned}\dot{X}_2 &= -\frac{\sqrt{130}Y_2}{10} + F_1(X_2,Y_2,Z_2),\\ \dot{Y}_2 &= \frac{\sqrt{130}X_2}{10} + F_2(X_2,Y_2,Z_2), \qquad (14)\\ \dot{Z}_2 &= \left(-\frac{3700}{169} + \frac{2000\sqrt{3}}{169}\right)Z_2 + F_3(X_2,Y_2,Z_2),\end{aligned}$$

where

$$F_1(X_2,Y_2,Z_2) = \frac{-\sqrt{130}(3781289+1956060\sqrt{3})}{161063152239444962}\Big(2504580000\sqrt{130}\sqrt{3}X_2Z_2 - 297034400\sqrt{3}{Y_2}^2 - 4633473000\sqrt{130}X_2Z_2 + 501222965600\sqrt{3}Z_2Y_2 - 46820000000\sqrt{3}{Z_2}^2 + 1375640565\,{X_2}^2 - 869742586000\,Z_2Y_2 + 82071914000\,{Z_2}^2 - 26\,(Y_2+Z_2)^2\big(148000000\,Z_2\sqrt{3} + 371293\,Y_2 - 256900000\,Z_2\big)\Big),$$

$$F_2(X_2,Y_2,Z_2) = \frac{257271293+148000000\sqrt{3}}{1238947324918807400}\Big(2504580000\sqrt{130}\sqrt{3}X_2Z_2 - 4633473000\sqrt{130}X_2Z_2 - 297034400\sqrt{3}{Y_2}^2 + 501222965600\sqrt{3}Z_2Y_2 - 46820000000\,{Z_2}^2\sqrt{3} + 1375640565\,{X_2}^2 - 869742586000\,Z_2Y_2 + 82071914000\,{Z_2}^2 - 26\,(Y_2+Z_2)^2\big(148000000\,Z_2\sqrt{3} + 371293\,Y_2 - 256900000\,Z_2\big)\Big),$$

$$F_3(X_2,Y_2,Z_2) = -\frac{257271293+148000000\sqrt{3}}{1238947324918807400}\Big(2504580000\sqrt{130}\sqrt{3}X_2Z_2 - 4633473000\sqrt{130}X_2Z_2 - 297034400\,{Y_2}^2\sqrt{3} + 501222965600\,Z_2Y_2\sqrt{3} - 46820000000\sqrt{3}{Z_2}^2 + 1375640565\,{X_2}^2 - 869742586000\,Z_2Y_2 + 82071914000\,{Z_2}^2 - 26\,(Y_2+Z_2)^2\big(148000000\,Z_2\sqrt{3} + 371293\,Y_2 - 256900000\,Z_2\big)\Big).$$

Next, the following quantities are evaluated at $a = a_1$, which followed a procedures proposed in (Hassard & Wan, 1978). According to that procedure, we obtain

$$g_{11} = \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial {X_2}^2} + \frac{\partial^2 F_1}{\partial {Y_2}^2} + i\left(\frac{\partial^2 F_2}{\partial {X_2}^2} + \frac{\partial^2 F_2}{\partial {Y_2}^2}\right)\right) = \frac{(-84443422000\sqrt{3}-186302634900)\sqrt{130}}{17351595881360} + i\left(\frac{856307414\sqrt{3}}{16684226809} + \frac{119598028113}{1334738144720}\right),$$

$$g_{02} = \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial {X_2}^2} - \frac{\partial^2 F_1}{\partial {Y_2}^2} - 2\frac{\partial^2 F_2}{\partial X_2 \partial Y_2} + i\left(\frac{\partial^2 F_2}{\partial {X_2}^2} - \frac{\partial^2 F_2}{\partial {Y_2}^2} + 2\frac{\partial^2 F_1}{\partial X_2 \partial Y_2}\right)\right) = \frac{(-205444670000\sqrt{3}-374084394900)\sqrt{130}}{17351595881360} + i\left(\frac{1885392586\sqrt{3}}{16684226809} + \frac{261678028113}{1334738144720}\right),$$

$$g_{20} = \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial {X_2}^2} - \frac{\partial^2 F_1}{\partial {Y_2}^2} + 2\frac{\partial^2 F_2}{\partial X_2 \partial Y_2} + i\left(\frac{\partial^2 F_2}{\partial {X_2}^2} - \frac{\partial^2 F_2}{\partial {Y_2}^2} - 2\frac{\partial^2 F_1}{\partial X_2 \partial Y_2}\right)\right) = \frac{(-205444670000\sqrt{3}-374084394900)\sqrt{130}}{17351595881360} + i\left(\frac{1885392586\sqrt{3}}{16684226809} + \frac{261678028113}{1334738144720}\right), \quad (15)$$

$$G_{21} = \frac{1}{8}\left(\frac{\partial^3 F_1}{\partial {X_2}^3} + \frac{\partial^3 F_1}{\partial X_2 \partial Y_2^2} + \frac{\partial^3 F_2}{\partial X_2^2 \partial Y_2} + \frac{\partial^3 F_2}{\partial {Y_2}^3} + i\left(\frac{\partial^3 F_2}{\partial {X_2}^3} + \frac{\partial^3 F_2}{\partial X_2 \partial Y_2^2} - \frac{\partial^3 F_1}{\partial X_2^2 \partial Y_2} - \frac{\partial^3 F_1}{\partial {Y_2}^3}\right)\right) = -\frac{14430000\sqrt{3}}{16684226809} - \frac{10033580427}{6673690723600} + i\,\frac{(-586818000\sqrt{3}-1134386700)\sqrt{130}}{6673690723600}.$$

Some calculations gives

$$h_{11} = \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial {X_2}^2} + \frac{\partial^2 F_3}{\partial {Y_2}^2}\right) = -\frac{119598028113}{1334738144720} - \frac{856307414\sqrt{3}}{16684226809}, \quad (16)$$

$$h_{20} = \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial {X_2}^2} - \frac{\partial^2 F_3}{\partial {Y_2}^2} - 2i\frac{\partial^2 F_3}{\partial X_2 \partial Y_2}\right) = -\frac{261678028113}{1334738144720} - \frac{1885392586\sqrt{3}}{16684226809}.$$

The solution of (9), here, is

$$\phi_{11} = -\frac{8535402627381}{133473814472000} - \frac{49266305077\sqrt{3}}{1334738144720},$$

$$\phi_{20} = -\frac{1694505419369269375\sqrt{3}}{76736658937506744962} - \frac{117422058874220264 25}{306946635750026979848} + i\sqrt{130}\Big(\frac{33645716739329585109}{613893271500053959 6960} + \frac{33645716739329585109}{6138932715000539596960} + \frac{121407729953773099\sqrt{3}}{38368329468753372481}\Big). \quad (17)$$

Let

$$G_{110} = \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial X_2 \partial Z_2} + \frac{\partial^2 F_2}{\partial Y_2 \partial Z_2} + i\left(\frac{\partial^2 F_2}{\partial X_2 \partial Z_2} - \frac{\partial^2 F_1}{\partial Y_2 \partial Z_2}\right)\right) = -\frac{15651753618\sqrt{3}}{216894948517} + \frac{140608808455}{216894948517} + i\,\frac{\sqrt{130}(558042995953700\sqrt{3}-1058831006128765)}{953036403783698},$$

$$G_{101} = \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial X_2 \partial Z_2} - \frac{\partial^2 F_2}{\partial Y_2 \partial Z_2} + i\left(\frac{\partial^2 F_2}{\partial X_2 \partial Z_2} + \frac{\partial^2 F_1}{\partial Y_2 \partial Z_2}\right)\right) = -\frac{55632446382\sqrt{3}}{216894948517} + \frac{353627076545}{216894948517} + i\,\frac{\sqrt{130}(-589887202859900\sqrt{3}+997272490905235)}{953036403783698},$$

$$g_{21} = G_{101}\phi_{20} + 2\,G_{110}\phi_{11} + G_{21} = -\frac{70583235606156203458777188079403\sqrt{3}}{1280291822278239489913326086258000} - \frac{61163254531067287069191247759 59}{6401459111391197449566630431290 0} + i\sqrt{130}\left(\frac{75657155784812542362023728134 57\sqrt{3}}{665751747584684534754929564854160} + \frac{26258053593715819427396984694767 1}{13315034951693690695098591297083200}\right). \quad (18)$$

Based on the above analysis, one can compute

$$C_1(0) = \frac{i}{2\omega}\left(g_{20}g_{11} - 2\,|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{1}{2}g_{21} = -\frac{407989171077528573662 7}{399030626475035073802400} - \frac{117534728144763059757 71\sqrt{3}}{1995153132375175369012000} + i\left(\frac{2070304325480359909322 99\sqrt{130}}{4149918515340364767544960 0} + \frac{300282510044430659852 9\sqrt{130}\sqrt{3}}{10374796288350911918862 40}\right),$$

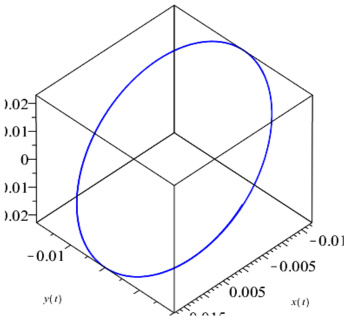
$$\mu_2 = -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(a_0))} = -\frac{9886888325047}{239166388136000} - \frac{4148585707813\sqrt{3}}{47833277627200},$$

$$\beta_2 = 2\,\mathrm{Re}(C_1(0)) = -\frac{407989171077528573662 7}{199515313237517536901200} - \frac{117534728144763059757 71\sqrt{3}}{997576566187587684506000}, \quad (19$$

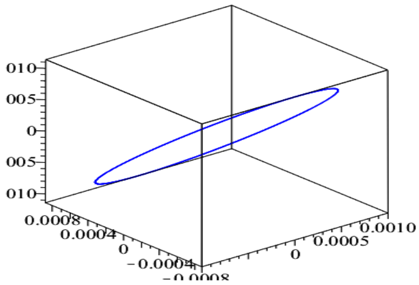
$$\tau_2 = -\frac{\mathrm{Im}(C_1(0)) + \mu_2\mathrm{Im}(\lambda'(a_0))}{\omega} = -\frac{15221689654211}{248733043661440} - \frac{218125707829\sqrt{3}}{6218326091536}$$

Since $\mu_2 < 0$, $\beta_2 < 0$ and $\tau_2 < 0$, then Hopf bifurcation is subcritical and non-degenrate with periodic orbit is unstable. Also the period of bifurcating periodic solutions decreases, which indicates that there is an orbitally unstable limit cycles. ■

It follows from Theorems 4.2 and 4.4 that three periodic orbits bifurcating from the origin and $(a, 0, 0)$ for the set of initial condition given by choosing initial condition$(0, 0.001, 0)$ and the periodic solutions occurs at the bifurcation values, performing numerical illustration using Maple software, as shown in Figure 1.

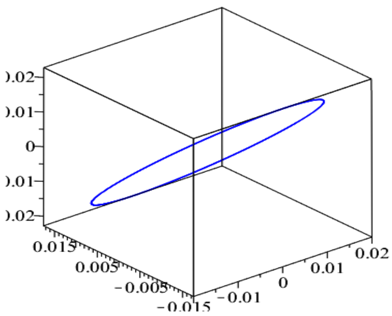


i.



(18)

ii.



iii.

Figure 1. The bifurcation periodic solution of system (1) for the values of the parameter i. when $a = \frac{13}{10}$ ii. $a = -\frac{370}{13} + \frac{200}{13}\sqrt{3}$ iii. $a = -\frac{370}{13} - \frac{200}{13}\sqrt{3}$.

4. CONCLUSIONS

In this work, local bifurcations and stability of cubic Jerk system (1) has been studied using Sotomayor's Theorem, Routh Hurwitz Criteria and Hopf bifurcation. We start by analyzing the transcritical bifurcation of system (1) at the equilibrium points and we proved that this system has only transcritical bifurcation at the origin when $a = 0$. Finally, it has proved that Hopf bifurcations occurs at equilibria points $O(0, 0, 0)$ and $E(a, 0, 0)$ when $a = \frac{13}{10}$ and $a = -\frac{370}{13} + \frac{200}{13}\sqrt{3}$ (or $a = -\frac{370}{13} - \frac{200}{13}\sqrt{3}$), in which these Hopf bifurcations are nondegenerate subcritical and supercritical (subcritical), respectively.

Moreover, we illustrated the results with a numerical example.

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