ZANCO Journal of Pure and Applied Sciences

TYPE Original Research
DOI http://dx.doi.org/10.21271/ZJPAS.36.2.9

# Existence Canard Solutions For Four Dimensional Hindmarsh-Rose Model with Respect to Infinitesimal Parameter

OPEN ACCESS

*CORRESPONDENCE

CHIMAN QADIR
chiman.qadir@su.edu.krd

RECEIVED :23 /07/ 2023
ACCEPTED :01/10/ 2023
PUBLISHED :15/ 04/ 2024

KEY WORDS:

Singularly perturbed dynamical systems, Hindmarsh-Rose model, Canard solutions, Nonstandard analysis.

Chiman Qadir [1], Ibrahim Hamad [1], Waleed Aziz [1] ,

[1] Department of Mathematics, College of Science, Salahaddin University–Erbil, Kurdistan region–Iraq

## ABSTRACT

This research endeavor seeks to investigate the presence and viability of canard solutions within the context of the generalized Hindmarsh-Rose model, specifically when extended to four dimensions. To achieve this, nonstandard analysis is employed as a powerful tool for identifying and characterizing canard solutions within the four-dimensional singularly perturbed system, wherein two fast variables are considered in the folded saddle case. By undertaking this rigorous approach, we aim to contribute valuable insights to the understanding of canard phenomena in complex dynamical systems.

## 1.Introduction

Multiscale systems refer to dynamical systems in which different variables or processes evolve on distinct time scales. Van der pol who was the first discovered and study of slow-fast dynamical system in 1920 and He was a pioneer in studying non-linear oscillations characterized by relaxation dynamics in 1926.(Van der Pol, 1960, Van der Pol, 1926)

Benoît and Lobry, Benoit (Benoît and Lobry, 1982, Benoît, 1983) introduced canard solutions within the context of three dimensional singularly perturbed systems with 2-slow and 1-fast variables and in (Benoît, 1983) the authors in a Nonstandard analysis version proved the existence of canard solutions. Later, the continuation works are in standard analysis version by Dumortier and Roussarie (Dumortier and Roussarie, 1996), they introduced and applied blow-up technique and Szmolyan and Wechselberger in (Szmolyan and Wechselberger, 2001) extended Geometric Singular Perturbation Theory to canards problems in $\mathbb{R}^3$ based on Benoît's theorem. Taking the progression of research a step further, Wechselberger in (Wechselberger, 2012) generalized Benoît's theorem to encompass n-dimensional singularly perturbed systems for both slow and fast variables. Ginoux and Llibre in (Ginoux and Llibre, 2016) presented a method to state a unique "generic" condition for the existence of canard solutions for three and four-dimensional singularly perturbed systems with only one fast. The collective contributions of these studies have significantly enhanced our understanding of canard phenomena in dynamical systems, both in three and four-dimensional contexts, and their insights have proven valuable for analyzing complex systems with singular perturbations.

The slow-fast dynamical system with four dimensions for studying canards solutions have three suitable forms. These forms include systems with a couple of slow and fast variables which is a standard once and the two other forms are with one slow and three fast variables and vice versa.

In (Tchizawa, 2007) the difficulties of slow-fast system in $\mathbb{R}^4$ to analyze in general. Given sufficient conditions make it possible to reduce the system into $\mathbb{R}^3$ with transversely conditions. In a subsequent research referenced in (Tchizawa, 2013) the existence of canard solutions in a multiple time scale $\mathbb{R}^{2+2}$ setting was investigated using two distinct approaches. The first approach involved an indirect method, where the original four-dimensional system was reduced to a 3-dimensional slow-fast system with two slow variables. The second approach, on the other hand, directly utilized the four-dimensional system and identified specific sufficient conditions to obtain the canard solutions. The article(Tchizawa, 2010) provides evidence for the existence of a relatively stable canard solution within a slow-fast system in $\mathbb{R}^{2+2}$, which is accompanied by an invariant manifold. The system has a four-dimensional canard solution with a relatively stable region. This stability is observed when an invariant manifold is present near the pseudo singular node point. The authors in (Tchizawa et al., 2005) a 4-dimensional system was analyzed to explore the presence of canard solutions in a two-region business cycle model, where each region is modeled using Goodwin's business cycle model, and they are coupled by interregional trade. They show that there exist canard solutions in the model with monotonic investment functions. Additionally, they provide results from numerical experiments to support their finding.

In (Corson and Aziz-Alaoui, 2009) they proceed the local stability and the numerical asymptotic analysis of Hindmarsh-Rose model are then developed in order to comprehend bifurcations and dynamics evolution of a single Hindmarsh-Rose neuron. The authors in (Ginoux et al., 2019) work for the extend method which improves the classical ones used to the case of three-dimensional singularly perturbed systems with two fast variables. This method state a unique generic condition for the existence of canard solutions for such three-dimensional singularly perturbed systems which is based on the stability of folded singularities of the normalized slow dynamics deduced from a well-known property of linear algebra. To demonstrate the existence of canard solutions, the Hindmarsh-Rose model is utilized as an example in this investigation. The researchers in (Hamad et al., 2023) investigated the presence of canard solutions in a three-dimensional system comprising slow and fast variables in $\mathbb{R}^{2+1}$. They explored alternative methods derived from Benoit's theorem to generate canard solutions and applied these methods to the Hindmarsh-Rose model, which involves two slow variables, to obtain canard solutions.

In (Desroches et al., 2013) they investigate the Mixed-mode bursting oscillations of the following form

$$\left.\begin{aligned} \varepsilon\dot{x} &= -ax^3 + bx^2 + y - z + I, \\ \varepsilon\dot{y} &= c - dx^2 - y, \\ \dot{z} &= s(x - x_e) - z, \\ \dot{I} &= k - h_x(x - x_{fold})^2 - h_y(y - y_{fold})^2 - h_I(I - I_{fold}), \end{aligned}\right\} \quad (1)$$

where $x, y, z, I$ are variables and $a, b, c, d, s, k, x_e, h_x, h_y, h_I, x_{fold}, y_{fold}, I_{fold}$ are parameters and $\varepsilon$ is infinitesimal in the sense of nonstandard analysis.

The system is an extension of the three dimensional Hindmarsh-Rose burster, where the main bifurcation parameter, the applied current $I$, evolves slowly. They study the appearance of orbits formed by a slow passage through a spike-adding transition.

The structure of this article is as follows: The second section provides an overview of the fundamental processes involved in obtaining canard solutions for four-dimensional singularly perturbed systems with two fast variables. In the third section, we focus on canard solutions in the extended Hindmarsh-Rose model when the two pseudo are of saddle types.

## 2. Background

A slow-fast system in $\mathbb{R}^{2+2}$ can be described as

$$\left.\begin{aligned} \varepsilon\frac{dx_1}{dt} &= \varphi_1(x_1, x_2, y_1, y_2, \varepsilon), \\ \varepsilon\frac{dx_2}{dt} &= \varphi_2(x_1, x_2, y_1, y_2, \varepsilon), \\ \frac{dy_1}{dt} &= \psi_1(x_1, x_2, y_1, y_2, \varepsilon), \\ \frac{dy_2}{dt} &= \psi_2(x_1, x_2, y_1, y_2, \varepsilon), \end{aligned}\right\} \quad (2)$$

where $\varphi = (\varphi_1, \varphi_2)$ and $\psi = (\psi_1, \psi_2)$ are standard defined on $\mathbb{R}^4 \times \mathbb{R}$ with $\varepsilon$ is infinitesimal. We begin with assumption of condition (I) to obtain an explicit solution.

(I) $\psi$ is of class $\mathbb{C}^1$ and $\varphi$ is of class $\mathbb{C}^2$.

Additionally, we suppose that (2) satisfies the generic conditions (II) up to (V).

(II) The set

$$S = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \mid \varphi_1(x_1, x_2, y_1, y_2, 0) = 0, \varphi_2(x_1, x_2, y_1, y_2, 0) = 0\},$$

is a 2-dimensional differentiable manifold, and the set

$$S_1 = \left\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \,\middle|\, \det\begin{bmatrix} \frac{\partial\varphi_1(x_1,x_2,y_1,y_2,0)}{\partial x_1} & \frac{\partial\varphi_1(x_1,x_2,y_1,y_2,0)}{\partial x_2} \\ \frac{\partial\varphi_2(x_1,x_2,y_1,y_2,0)}{\partial x_1} & \frac{\partial\varphi_2(x_1,x_2,y_1,y_2,0)}{\partial x_2} \end{bmatrix} = 0\right\}$$

,

which is a 3-dimensional differentiable manifold, whereas the generalized pli set $GPL = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \mid S \cap S_1\}$ is a 1-dimensional differentiable manifold.

(III) The values of $\psi_1 \neq 0$ or $\psi_2 \neq 0$ at any point $p \in GPL$.

In (2), when $\varepsilon = 0$ then the algebraic-differential equations is obtained as follows

$$\left.\begin{aligned} 0 &= \varphi_1(x_1,x_2,y_1,y_2,0),\\ 0 &= \varphi_2(x_1,x_2,y_1,y_2,0),\\ \frac{dy_1}{dt} &= \psi_1(x_1,x_2,y_1,y_2,0),\\ \frac{dy_2}{dt} &= \psi_2(x_1,x_2,y_1,y_2,0), \end{aligned}\right\} \tag{3}$$

Differentiating both equations $\varphi_1$ and $\varphi_2$ in (3) with respect to $t$, and for simplicity we use the notations $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then we obtain

$$\left.\begin{aligned} \frac{\partial\varphi_1(x,y,0)}{\partial x_1}\dot{x}_1 + \frac{\partial\varphi_1(x,y,0)}{\partial x_2}\dot{x}_2 + \frac{\partial\varphi_1(x,y,0)}{\partial y_1}\dot{y}_1 + \frac{\partial\varphi_1(x,y,0)}{\partial y_2}\dot{y}_2 = 0,\\ \frac{\partial\varphi_2(x,y,0)}{\partial x_1}\dot{x}_1 + \frac{\partial\varphi_2(x,y,0)}{\partial x_2}\dot{x}_2 + \frac{\partial\varphi_2(x,y,0)}{\partial y_1}\dot{y}_1 + \frac{\partial\varphi_2(x,y,0)}{\partial y_2}\dot{y}_2 = 0. \end{aligned}\right\} \tag{4}$$

the matrix form of (4) is given as follows

$$\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial x_1} & \frac{\partial\varphi_1(x,y,0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial x_1} & \frac{\partial\varphi_2(x,y,0)}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1\\ \dot{x}_2 \end{bmatrix} = -\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial y_1} & \frac{\partial\varphi_1(x,y,0)}{\partial y_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial y_1} & \frac{\partial\varphi_2(x,y,0)}{\partial y_2} \end{bmatrix} \begin{bmatrix} \dot{y}_1\\ \dot{y}_2 \end{bmatrix}. \tag{5}$$

Multiply both sides of (5) by the inverse of the matrix $\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial x_1} & \frac{\partial\varphi_1(x,y,0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial x_1} & \frac{\partial\varphi_2(x,y,0)}{\partial x_2} \end{bmatrix}$ and then we put the result in (3) then the algebraic-differential equations become

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}_1\\ \dot{x}_2 \end{bmatrix} &= -\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial x_1} & \frac{\partial\varphi_1(x,y,0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial x_1} & \frac{\partial\varphi_2(x,y,0)}{\partial x_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial y_1} & \frac{\partial\varphi_1(x,y,0)}{\partial y_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial y_1} & \frac{\partial\varphi_2(x,y,0)}{\partial y_2} \end{bmatrix} \begin{bmatrix} \psi_1(x,y,0)\\ \psi_2(x,y,0) \end{bmatrix},\\ \begin{bmatrix} \dot{y}_1\\ \dot{y}_2 \end{bmatrix} &= \begin{bmatrix} \psi_1(x,y,0)\\ \psi_2(x,y,0) \end{bmatrix}, \end{aligned}\right\} \tag{6}$$

for any $(x, y) \in S\backslash GPL$. In order to avoid the degeneracy in equation (6), we use the rescaling $\det\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial x_1} & \frac{\partial\varphi_1(x,y,0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial x_1} & \frac{\partial\varphi_2(x,y,0)}{\partial x_2} \end{bmatrix}$. Then the time-scaled reduced system is given as follows

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}_1\\ \dot{x}_2 \end{bmatrix} &= -adj\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial x_1} & \frac{\partial\varphi_1(x,y,0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial x_1} & \frac{\partial\varphi_2(x,y,0)}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial y_1} & \frac{\partial\varphi_1(x,y,0)}{\partial y_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial y_1} & \frac{\partial\varphi_2(x,y,0)}{\partial y_2} \end{bmatrix} \begin{bmatrix} \psi_1(x,y,0)\\ \psi_2(x,y,0) \end{bmatrix},\\ \begin{bmatrix} \dot{y}_1\\ \dot{y}_2 \end{bmatrix} &= det\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial x_1} & \frac{\partial\varphi_1(x,y,0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial x_1} & \frac{\partial\varphi_2(x,y,0)}{\partial x_2} \end{bmatrix} \begin{bmatrix} \psi_1(x,y,0)\\ \psi_2(x,y,0) \end{bmatrix}. \end{aligned}\right\} \tag{7}$$

(IV) The rank $\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial x_1} & \frac{\partial\varphi_1(x,y,0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial x_1} & \frac{\partial\varphi_2(x,y,0)}{\partial x_2} \end{bmatrix} = 2$ for any $(x, y)$ in $S\backslash\, GPL$, and the

rank $\begin{bmatrix} \frac{\partial\varphi_1(x,y,0)}{\partial y_1} & \frac{\partial\varphi_1(x,y,0)}{\partial y_2}\\ \frac{\partial\varphi_2(x,y,0)}{\partial y_1} & \frac{\partial\varphi_2(x,y,0)}{\partial y_2} \end{bmatrix} = 2$ for any $(x, y) \in S$.

Then, the surface $S$ can be write as $y = h(x)$ in the neighborhood of $GPL$.

Let $y = h(x)$ with $h = (h_1, h_2)$. On the set $S$, differentiating the equation $y = h(x)$ with respect to $t$, we can get

$$\begin{bmatrix}\dot{y}_1\\ \\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}\frac{\partial h_1(x,0)}{\partial x_1} & \frac{\partial h_1(x,0)}{\partial x_2}\\ \frac{\partial h_2(x,0)}{\partial x_1} & \frac{\partial h_2(x,0)}{\partial x_2}\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \\ \dot{x}_2\end{bmatrix}. \tag{8}$$

From the two equations (5) and (8), we conclude that

$$\begin{bmatrix}\frac{\partial\varphi_1(x,h(x),0)}{\partial x_1} & \frac{\partial\varphi_1(x,h(x),0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,h(x),0)}{\partial x_1} & \frac{\partial\varphi_2(x,h(x),0)}{\partial x_2}\end{bmatrix}+\begin{bmatrix}\frac{\partial\varphi_1(x,h(x),0)}{\partial y_1} & \frac{\partial\varphi_1(x,h(x),0)}{\partial y_2}\\ \frac{\partial\varphi_2(x,h(x),0)}{\partial y_1} & \frac{\partial\varphi_2(x,h(x),0)}{\partial y_2}\end{bmatrix}\begin{bmatrix}\frac{\partial h_1(x,0)}{\partial x_1} & \frac{\partial h_1(x,0)}{\partial x_2}\\ \frac{\partial h_2(x,0)}{\partial x_1} & \frac{\partial h_2(x,0)}{\partial x_2}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}. \tag{9}$$

Thus the following is obtained from equation above

$$\begin{bmatrix}\frac{\partial h_1(x,0)}{\partial x_1} & \frac{\partial h_1(x,0)}{\partial x_2}\\ \frac{\partial h_2(x,0)}{\partial x_1} & \frac{\partial h_2(x,0)}{\partial x_2}\end{bmatrix}=-\begin{bmatrix}\frac{\partial\varphi_1(x,h(x),0)}{\partial y_1} & \frac{\partial\varphi_1(x,h(x),0)}{\partial y_2}\\ \frac{\partial\varphi_2(x,h(x),0)}{\partial y_1} & \frac{\partial\varphi_2(x,h(x),0)}{\partial y_2}\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial\varphi_1(x,h(x),0)}{\partial x_1} & \frac{\partial\varphi_1(x,h(x),0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,h(x),0)}{\partial x_1} & \frac{\partial\varphi_2(x,h(x),0)}{\partial x_2}\end{bmatrix}. \tag{10}$$

Since, $\begin{bmatrix}\dot{y}_1\\ \\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}\psi_1(x,h(x),0)\\ \\ \psi_2(x,h(x),0)\end{bmatrix}$ and from equation (8) the slow system can be simplified to the following

$$\begin{bmatrix}\frac{\partial h_1(x,0)}{\partial x_1} & \frac{\partial h_1(x,0)}{\partial x_2}\\ \frac{\partial h_2(x,0)}{\partial x_1} & \frac{\partial h_2(x,0)}{\partial x_2}\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\psi_1(x,h(x),0)\\ \\ \psi_2(x,h(x),0)\end{bmatrix}. \tag{11}$$

Put the equation (10) in the system above reduces to

$$\begin{bmatrix}\frac{\partial\varphi_1(x,h(x),0)}{\partial x_1} & \frac{\partial\varphi_1(x,h(x),0)}{\partial x_2}\\ \frac{\partial\varphi_2(x,h(x),0)}{\partial x_1} & \frac{\partial\varphi_2(x,h(x),0)}{\partial x_2}\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \\ \dot{x}_2\end{bmatrix}=-\begin{bmatrix}\frac{\partial\varphi_1(x,h(x),0)}{\partial y_1} & \frac{\partial\varphi_1(x,h(x),0)}{\partial y_2}\\ \frac{\partial\varphi_2(x,h(x),0)}{\partial y_1} & \frac{\partial\varphi_2(x,h(x),0)}{\partial y_2}\end{bmatrix}\begin{bmatrix}\psi_1(x,h(x),0)\\ \\ \psi_2(x,h(x),0)\end{bmatrix}. \tag{12}$$

Then the projected system is obtained

$$\begin{bmatrix}\dot{x}_1\\ \\ \dot{x}_2\end{bmatrix}=-\begin{bmatrix}\frac{\partial\varphi_2(x,h(x),0)}{\partial x_1} & -\frac{\partial\varphi_1(x,h(x),0)}{\partial x_2}\\ -\frac{\partial\varphi_2(x,h(x),0)}{\partial x_1} & \frac{\partial\varphi_1(x,h(x),0)}{\partial x_1}\end{bmatrix}\begin{bmatrix}\frac{\partial\varphi_1(x,h(x),0)}{\partial y_1} & \frac{\partial\varphi_1(x,h(x),0)}{\partial y_2}\\ \frac{\partial\varphi_2(x,h(x),0)}{\partial y_1} & \frac{\partial\varphi_2(x,h(x),0)}{\partial y_2}\end{bmatrix}\begin{bmatrix}\psi_1(x,h(x),0)\\ \\ \psi_2(x,h(x),0)\end{bmatrix} \tag{13}$$

(V) The system (13) is the time scaled reduced system projected into $\mathbb{R}^2$.

Again, we suppose the set $S_1$ is nonempty. All the singular points of the system (13) are

nondegenerate, that is, the obtained matrix from the linearized system of (13) at a singular

point has distinct non-zero eigenvalues.

**Definition 2.1**. Let $p \in GPS$ and $\lambda_1$, $\lambda_2$ be two eigenvalues of the Jacobean of system (13) at $p \in \mathbb{R}^4$ . If two real eigenvalues have opposite signs, then the point $p$ is called a generalized pseudo singular saddle, and if both are either negative or positive, it is called a generalized pseudo singular node.

**Remark 2.1**. In equation (13) the set $\{(x,y) \in GPL| \begin{bmatrix}\dot{x}_1\\ \\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0\\ \\ 0\end{bmatrix}\}$ is defined as a set of generalized pseudo singular points and denoted by $GPS$. This approach is called a direct method that transforms the original system to the time scaled reduced system directly.

Now, we provide an explanation of the definition of a canard in $\mathbb{R}^4$ along the direct method.

**Definition 2.2.** Let a point $p$ be in $GPS$. A canard solution in $\mathbb{R}^4$ refers to a trajectory that start by following the attractive surface and the saddle point, and then continuous along a slow manifold, which is non-infinitesimal. Additionally, we suppose the following.

(VI) The invariant manifold $Inv(\varphi(x,y))$ lying near the $GPS$ has 2-dimension in $\mathbb{R}^4$.
It intersects $GPS$ transversely.

**Theorem 2.3**. Suppose the origin be the generalized pseudo singularity saddle or node. Then there exists a canard solution in $\mathbb{R}^4$. If a local model satisfies the following conditions

1) The matrix $[\varphi_x(0,h(x),0)]$ has two different eigenvalues which are zero and negative values.
2) A local model satisfies

***First***: The trace of the matrix $[\varphi_x(0,h(0),0)]$ less than zero.

***Second:*** $\psi_1(O) \neq 0$, $\psi_2(O) \neq 0$, where $O = (0,0,0,0)$.

**Theorem 2.4**. If the system has a square-linear solution in a local model, for any $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}$, there exist essentially two local models describing the explicit duck solutions.

1) For $\alpha_1 = 2,\ \alpha_2 = 1,\ \alpha_3 = 2,\ \alpha_4 = 2$ with the explicit duck solution,
when $\delta = 0$,
$$u_1 = -\frac{\left(h_{1_{y_1}}(0)f_1(0) + h_{1_{y_2}}(0)f_2(0)\right)t}{h_{1_{x_1}}(0)}, u_2 = 0, v_1 = f_1(0)t, v_2 = f_2(0)t.$$
And for $\delta \neq 0$,
$$u_1 = -\frac{\left(h_{1_{y_1}}(0)f_1(0) + h_{1_{y_2}}(0)f_2(0)\right)t}{h_{1_{x_1}}(0)} + L(\delta), u_2 = -L(\delta),$$
$$v_1 = f_1(0)t + L(\delta), v_2 = f_2(0)t + L(\delta).$$

2) For $\alpha_1 = 2,\ \alpha_2 = 1,\ \alpha_3 = 3,\ \alpha_4 = 2$ with the explicit duck solution,
when $\delta = 0$,
$$u_1 = -\frac{h_{1_{y_2}}(0)f_2(0)t}{h_{1_{x_1}}(0)}, u_2 = 0, v_1 = constant, v_2 = f_2(0)t.$$
And for $\delta \neq 0$,
$$u_1 = -\frac{h_{1_{y_2}}(0)f_2(0)t}{h_{1_{x_1}}(0)} + L(\delta), u_2 = -L(\delta),$$
$$v_1 = constant + L(\delta), v_2 = f_2(0)t + L(\delta).$$

All background section can be found in (Tchizawa, 2013, Tchizawa, 2012)

Assume the origin belongs to the generalized pseudo singularity, which must either be a saddle or node. The change of variables $x_1 = r^{\alpha_1} u_1$, $x_2 = r^{\alpha_2} u_2$, $y_1 = r^{\alpha_3} v_1$, $y_2 = r^{\alpha_4} v_2$ correspond to microscopes $r$ is infinitesimal, the original system transforms to the system with new variables $u_1, u_2, v_1, v_2$. Then there exist local models that provide an explanation for the 4-dimensional canard solutions.

## 3. Results

The system (2) yields the following outcomes for the pseudo singular saddle or node.

**Proposition 3.1**. If the system (2) in a local model has a square-linear solution for any value of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}$, then there is essentially a local model that describes the explicit canard solution.

**Proof.** In order to prove our argument, we utilize the conditions on

$\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 2,\ \alpha_4 = 2,$

and make changes to the variables
$x_1 = ru_1,\ x_2 = r^2 u_2,\ y_1 = r^2 v_1,\ y_2 = r^2 v_2.$

In order to convert the main system into

$$\left.\begin{aligned} \varepsilon r\, \dot{u}_1 &= \varphi_1(ru_1,\ r^2u_2,\ r^2 v_1,\ r^2 v_2), \\ \varepsilon r^2 \dot{u}_2 &= \varphi_2(ru_1,\ r^2u_2,\ r^2 v_1,\ r^2 v_2), \\ r^2\, \dot{v}_1 &= \psi_1(ru_1,\ r^2u_2,\ r^2 v_1,\ r^2 v_2), \\ r^2\, \dot{v}_2 &= \psi_2(ru_1,\ r^2u_2,\ r^2 v_1,\ r^2 v_2). \end{aligned}\right\} \quad (14)$$

In above system, multiply the second equation by $r^2$ and the first and forth ones by $r$ and rescaling the system by $t = r^2\tau$ then $dt = r^2 d\tau$ to obtain the following:

$$\left.\begin{aligned} \frac{\varepsilon}{r^2}\, {u_1}' &= \frac{\varphi_1(ru_1,\ r^2u_2,\ r^2\, v_1,\ r^2\, v_2)}{r}, \\ \frac{\varepsilon}{r^2} {u_2}' &= \frac{\varphi_2(ru_1,\ r^2u_2,\ r^2\, v_1,\ r^2\, v_2)}{r^2}, \\ {v_1}' &= \psi_1(ru_1,\ r^2u_2,\ r^2\, v_1,\ r^2\, v_2), \\ {v_2}' &= \psi_2(ru_1,\ r^2u_2,\ r^2\, v_1,\ r^2\, v_2). \end{aligned}\right\} \quad (15)$$

Both systems (14) and (15) are equivalent. With the use of assumptions (I) and (IV), we develop a local model that operates in the simplest possible conditions.

(1) $trace[\varphi(0, h(0), 0)] < 0,$
(2) $\psi_1(O) \neq 0,\ \psi_2(O) \neq 0,$
where $O = (0, 0, 0)$, put $\frac{\varepsilon}{r^2} = \delta$ infinitesimal, then

$$\left.\begin{aligned} \delta u_1' &= \varphi_{1_{x_1}}\, u_1 + O(ru_1, ru_2, rv_1, rv_2), \\ \delta u_2' &= \frac{\varphi_{2_{x_1}}}{r} u_1 + \frac{\varphi_{2_{x_1 x_1}}}{2}\, u_1^2 + \varphi_{2_{x_2}} u_2 + \varphi_{2_{y_1}} v_1 + \varphi_{2_{y_2}} v_2 + O(ru_1, ru_2, rv_1, rv_2), \\ v_1' &= \psi_1 + O(ru_1, ru_2, rv_1, rv_2), \\ v_2' &= \psi_2 + O(ru_1, ru_2, rv_1, rv_2). \end{aligned}\right\} \quad (16)$$

Let's assume again that the $trace\ [\varphi(0, h(0), 0)]$ is less than zero, due to the fact that the fast vector field has one zero eigenvalue and the other one is negative. The solution in the local model is

$$u_1(\tau) \simeq 0,\ u_2(\tau) \simeq -\frac{\psi_1\, \varphi_{2_{y_1}} + \psi_2 \varphi_{2_{y_2}}}{\varphi_{2_{x_2}}}\, \tau,\ v_1(\tau) \simeq \psi_1 \tau,\ v_2(\tau) \simeq \psi_2 \tau,$$

Our previous result and Proposition (3.1) and Theorem (2.4) apply to the generalized Hindmarsh-Rose model in four dimensional system.
In system (1), we suppose that $\omega = y + I$ then the system reduces to

$$\left.\begin{aligned} \varepsilon\dot{x} &= -ax^3 + bx^2 + \omega - z, \\ \varepsilon\dot{\omega} &= c - dx^2 - \omega + I + \varepsilon\left(k - h_x\left(x - x_{fold}\right)^2 - h_y\left(\omega - I - y_{fold}\right)^2 - h_I\left(I - I_{fold}\right)\right), \\ \dot{z} &= s(x - x_e) - z, \\ \dot{I} &= k - h_x\left(x - x_{fold}\right)^2 - h_y\left(\omega - I - y_{fold}\right)^2 - h_I\left(I - I_{fold}\right), \end{aligned}\right\} \quad (17)$$

Now, let's make the following variable changes in (17) in order to apply the procedure described in previous section:

$$x \to x_1,\ \omega \to x_2,\ z \to y_1,\ I \to y_2.$$

Therefore, we have

$$\left.\begin{aligned} \varepsilon\dot{x}_1 &= -a{x_1}^3 + b{x_1}^2 + x_2 - y_1, \\ \varepsilon\dot{x}_2 &= c - d{x_1}^2 - x_2 + y_2 + \varepsilon\left(k - h_x\left(x_1 - x_{fold}\right)^2 - h_y\left(x_2 - y_2 - y_{fold}\right)^2 - h_I\left(y_2 - I_{fold}\right)\right), \\ \dot{y}_1 &= s(x_1 - x_e) - y_1, \\ \dot{y}_2 &= k - h_x\left(x_1 - x_{fold}\right)^2 - h_y\left(x_2 - y_2 - y_{fold}\right)^2 - h_I\left(y_2 - I_{fold}\right). \end{aligned}\right\} \quad (18)$$

When $\varepsilon = 0$, the time-scaled reduced system is obtained from the system (18) as follows

$$\left.\begin{array}{l}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 2dx_1 & -3ax_1^2+2bx_1\end{bmatrix}\begin{bmatrix}s(x_1-x_e)-y_1\\ k-h_x\left(x_1-x_{fold}\right)^2-h_y\left(x_2-y_2-y_{fold}\right)^2-h_I(y_2-I_{fold})\end{bmatrix},\\ \begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=det\begin{bmatrix}-3ax_1^2+2bx_1 & 1\\ -2dx_1 & -1\end{bmatrix}\begin{bmatrix}s(x_1-x_e)-y_1\\ k-h_x\left(x_1-x_{fold}\right)^2-h_y\left(x_2-y_2-y_{fold}\right)^2-h_I(y_2-I_{fold})\end{bmatrix}\end{array}\right\} \quad (19)$$

the system (18) has pseudo singular points, when we solve $\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ in (19) and $x_1\big(3ax_1-2(b-d)\big)=0$, then we can get the following pseudo singular points

$$x_1 = 0,$$
$$x_2=\frac{k+sx_e-h_xx_{fold}^2+h_I\left(I_{fold}+c\right)+h_y\left(y_{fold}-c\right)}{h_I-1},$$
$$y_1 = x_2,$$
$$y_2 = x_2 - c,$$

with $h_I \neq 1$ and the other pseudo singular point is

$$x_1=\frac{2(b-d)}{3a},$$
$$x_2=y_1-\frac{4}{27a^2}(b+2d)(b-d)^2,$$
$$y_1=\frac{27a^3s\big(3ax_e-2(b-d)\big)+h_xl_1-h_yl_2^2+81a^4k+3a^2hI\left(27a^2\left(c+I_{fold}\right)+4(b-d)^3\right)}{81a^4\,(hI-1)},$$
$$y_2= y_1-c-\frac{4}{27a^2}(b-d)^3,$$

where $l_1=-9a^2(3a\,x_{fold}-2(b-d))^2,\ l_2=9a^2\left(c-y_{fold}\right)-4d(b-d)^2$ ,
with $a\neq 0,\ h_I\neq 1, b\neq d$.

Now, to find an explicit canard solution according to use the information above for each pseudo singular point separately.

**3.1. First pseudo singular point**. In this subsection focuses on saddle pseudo singular point and determine the existence of canard solutions.

The system (18) is transformed into the origin by the change of variable

$$x_1 = x_{11},$$
$$x_2=x_{22}+\frac{k+sx_e-h_xx_{fold}^2+h_I\left(I_{fold}+c\right)+h_y\left(y_{fold}-c\right)}{h_I-1},$$
$$y_1=y_{11}+\frac{k+sx_e-h_xx_{fold}^2+h_I\left(I_{fold}+c\right)+h_y\left(y_{fold}-c\right)}{h_I-1},$$
$$y_2=y_{22}+\frac{k+sx_e-h_xx_{fold}^2+h_I\,I_{fold}+c+h_y\left(y_{fold}-c\right)}{h_I-1},$$

The obtained system is as follows

$$\left.\begin{array}{l}\varepsilon\,\dot{x}_{11}=-ax_{11}^3+bx_{11}^2+x_{22}-y_{11},\\ \varepsilon\,\dot{x}_{22}=-dx_{11}^2-x_{22}+y_{22}-\varepsilon(h_x\left(x_{11}-2x_{fold}\right)x_{11}-h_y\left(x_{22}-y_{22}+c-y_{fold}\right)^2-h_Iy_{22}+l_3),\\ \dot{y}_{11}=sx_{11}-y_{11}+l_4,\\ \dot{y_{22}}=-h_x\left(x_{11}-2x_{fold}\right)x_{11}-h_y\left(x_{22}-y_{22}+c-y_{fold}\right)^2-h_Iy_{22}+l_3\end{array}\right\} \quad (20)$$

where $l_3 = -\frac{k-h_x x_{fold}^2 + h_I(c(h_y+1)+h_y y_{fold}+s x_e+I_{fold})}{h_I-1}$, $l_4 = -\frac{k-h_x x_{fold}^2+h_I(c+s x_e+I_{fold})-h_y(c-y_{fold})}{h_I-1}$

By using the equation (13) then the projected system of (20) according to the $y_{11} = -a x_{11}^3 + b x_{11}^2 + x_{22},\ y_{22} = d x_{11}^2 + x_{22}$ is

$$\begin{bmatrix}\dot{x}_{11}\\ \dot{x}_{22}\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 2d x_{11} & x_{11}(-3a x_{11}+2b)\end{bmatrix}\begin{bmatrix}s x_1 + a x_1^3 - b x_1^2 - x_2 - l_4\\ l_5\end{bmatrix}, \tag{21}$$

where

$$l_5 = -(h_x(x_{11} - 2x_{fold})x_{11} - h_y\left(x_{22} - y_{22} + c - y_{fold}\right)^2 - h_I y_{22} + l_3).$$

The Jacobian matrix of system (21) has two eigenvalues of the form

$$\lambda_1 = -2(x_{fold} + h_I x_e)(b-d),\ \lambda_2 = 2(h_I x_e - x_{fold})(b-d) - s, \tag{22}$$

under the conditions on parameters

$$c = y_{fold},\ k = s x_{fold}, h_x = -2(b-d),\ I_{fold} = -2x_e^2 h_I(b-d) - y_{fold}. \tag{23}$$

If $(x_{fold} + h_I x_e)(b-d) > 0$ and $2(x_{fold} - h_I x_e)(b-d) - s > 0$ or $(x_{fold} + h_I x_e)(b-d) < 0$ and $2(x_{fold} - h_I x_e)(b-d) - s < 0$, then the Jacobian matrix has two real opposite eigenvalues, and the generalized pseudo singular point is saddle. The matrix $\left[\frac{\partial\varphi}{\partial X}\right] = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}$ at origin has the zero and negative eigenvalues with the negative trace and $\psi_1 = l_4, \psi_2 = l_3$ , where $X = (x_{11}, x_{22})$ then the local model (20) has an explicit canard solution by Theorem (2.3).

By using Proposition (3.1) to find explicit canard solution for the system (20).
Using the change of variables $x_{11} = r u_1,\ x_{22} = r^2 u_2,\ y_{11} = r^2 v_1,\ y_{22} = r^2 v_2$, rescaling $t = r^2\tau$ and $\frac{\varepsilon}{r^2} = \delta$ and we put the conditions (18) in (15) then the obtained system is

$$\left.\begin{aligned}\delta u_1' &= O(r u_1, r u_2, r v_1, r v_2),\\ \delta u_2' &= -\frac{d}{2}u_1^2 - u_2 + v_2 + O(r u_1, r u_2, r v_1, r v_2),\\ v_1' &= l_6 + O(r u_1, r u_2, r v_1, r v_2),\\ v_2' &= l_6 + O(r u_1, r u_2, r v_1, r v_2).\end{aligned}\right\} \tag{24}$$

where $l_6 = \frac{(h_I x_e + x_{fold})(2(h_I x_e - x_{fold})(b-d) - s)}{h_I - 1}$.

We can get the solution

$$u_1 \simeq 0,\ u_2 \simeq l_6\,\tau,\ v_1 \simeq l_6\,\tau,\ v_2 \simeq l_6\,\tau,$$

Reversing the solution to the original variables as follows

$$x \simeq 0,\ y \simeq 0,\ z \simeq l_6\, t,\ I \simeq l_6\, t, \tag{25}$$

**3.2. Second pseudo singular point**. We determine the existence of canard solutions as in the previous subsection.
The change of coordinates converts the system (18) to the origin

$$x_1 = x_{11} + \frac{2(b-d)}{3a},$$

$$x_2 = x_{22} + y_1 - \frac{4}{27a^2}(b+2d)(b-d)^2,$$

$$y_1 = y_{11} + \frac{27a^3 s(3a x_e - 2(b-d)) + h_x l_1 - h_y l_2^2 + 81a^4 k + 3a^2 hI\,(27a^2(c+I_{fold}) + 4(b-d)^3)}{81a^4\,(hI-1)},$$

$$y_2 = y_{22} + y_1 - c - \frac{4}{27a^2}(b - d)^3,$$

The system takes the following form

$$\left.\begin{aligned} \varepsilon \dot{x}_{11} &= -\frac{3a^2 x_{11}^3 + 3a(b-2d)x_{11}^2 - 4d(b-d)x_{11} - 3ax_{22} + 3ay_{11}}{3a}, \\ \varepsilon \dot{x}_{22} &= -\frac{3adx_{11}^2 + 4d(b-d)x_{11} + 3ax_{22} - 3ay_{22}}{3a} + \varepsilon \dot{y}_{22}, \\ \dot{y}_{11} &= sx_{11} - y_{11} - m_2, \\ \dot{y}_{22} &= -h_y(y_{22} - x_{22})^2 - y_{22}h_I - h_y m_1(x_{11} - y_{22}) - h_x x_{11}\left(x_{11} - \frac{2\left(3ax_{fold} - 2(b-d)\right)}{3a}\right) - m_2, \end{aligned}\right\} \quad (26)$$

where $m_1 = \frac{18a^2(c - y_{fold}) - 8d(b-d)^2}{9a^2}, m_2 = \left(y_1 + \frac{s\left(3ax_e - 2(b-d)\right)}{3a}\right).$

By using the equation (13) then the projected system of (26) according to

$$y_{11} = -\left(3a^2 x_{11}^3 + 3a(b - 2d)x_{11}^2 - 4d(b - d)x_{11} - 3ax_{22}\right),$$
$$y_{22} = 3adx_{11}^2 + 4d(b - d)x_{11} + 3ax_{22}$$

is as follows

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{22} \end{bmatrix} = -\begin{bmatrix} -1 & -1 \\ 2dx_{11} + \frac{4d(b-d)}{3a} & x_{11}\left(-3ax_{11} - 2(b-2d)\right) + \frac{4d(b-d)}{3a} \end{bmatrix} \begin{bmatrix} -ax_{11}^3 - (b-2d)x_{11}^2 + x_{22} - (3as - 4d(b-d)x_{11} + m_2 \\ -h_I x_{22} - d^2 h_y x_{11}^4 - \frac{8(b-d)d^2 h_y x_{11}^3}{3a} + l_7 \end{bmatrix}, \quad (27)$$

where

$$l_7 = \left(\left(2d\left(c - y_{fold}\right) - \frac{8d^2(b-d)^2}{3a^2}\right)h_y - \left(dh_I + h_x\right)\right)x_{11}^2 - \frac{-6a(h_I - 1)(4d(b-d)m_2 + 9a^2\left(3ah_x x_{fold} - 2(dh_I + h_x)(b-d)\right)}{81a^4(h_I - 1)}x_{11} + y_1.$$

The Jacobian matrix of system (27) possesses two eigenvalues of the form

$$\lambda_1 = -\frac{4h_I d(b-d)}{3a(27a^2+1)}, \quad \lambda_2 = \frac{4d(h_I - 1)(b-d)}{3a}, \quad (28)$$

subject to the specified parameter conditions,

$$I_{fold} = -\frac{2(b-d)^2(2b-5d)}{27a^2}, \quad c = \frac{8h_I d^2(b-d)}{243a^4}, \quad s = \frac{4d(h_y+1)(b-d)}{3a}, x_{fold} = \frac{2(b-d)}{3a},$$
$$x_e = x_{fold}, \quad y_{fold} = -\frac{(27a^2+1)\left(216d(b-d)^2a^2 + 243a^4 - 16d^2h_I(b-d)\right)h_y + 6561a^6h_I}{486a^4h_y(27a^2+1)}$$
$$k = \frac{243a^4h_y^2(27a^2+1)^2 + (-32d^2(27a^2+1)(b-d)h_I^2 + 54a^2(27a^2+1)(-4d(b-d)(27a^2(b-d)+b+3d) + 243a^4)h_I)h_y + 177147a^8h_I^2}{972a^4h_y(27a^2+1)^2},$$

If $1 < h_I$ or $h_I < 0$ then the Jacobian matrix has two real opposite eigenvalues, and the generalized pseudo singular point is saddle.

The matrix $\left[\frac{\partial\varphi}{\partial X}\right] = \begin{bmatrix} \frac{4d(b-d)}{3a} & 1 \\ -\frac{4d(b-d)}{3a} & -1 \end{bmatrix}$ in system (27) at the origin has the zero and negative eigenvalues with the condition $\frac{4d(b-d)}{3a} - 1 < 0$ and $\psi_1 = \psi_2 = -m_2$, then the local model (26) has an explicit canard solution by Theorem (2.3).

Using Theorem (2.4), to find explicit canard solution for the system (26), we use the change of variables $x_{11} = r^2 u_1$, $x_{22} = r\, u_2$, $y_{11} = r^2 v_1$, $y_{22} = r^2 v_2$, rescaling $t = r^2 \tau$ and $\frac{\varepsilon}{r^2} = \delta$, where $\delta$ infinitesimal in (27), the system that is derived from this process is

$$\left.\begin{aligned}\delta u_1' &= \frac{4d(b-d)}{3a}u_1 - v_1 + \frac{u_2}{r} + O(ru_1, ru_2, rv_1, rv_2),\\ \delta u_2' &= -u_2 + O(ru_1, ru_2, rv_1, rv_2),\\ v_1' &= -\frac{8d^2 h_I (b-d)}{9a^2(27a^2+1)} + O(ru_1, ru_2, rv_1, rv_2)\\ v_2' &= -\frac{8d^2 h_I (b-d)}{9a^2(27a^2+1)} + O(ru_1, ru_2, rv_1, rv_2).\end{aligned}\right\} \quad (29)$$

The solution can be obtained in the following manner,

$$u_1 \simeq -\frac{2dh_I}{3a(27a^2+1)}\tau,\ u_2 \simeq 0, v_1 \simeq -\frac{8d^2 h_I (b-d)}{9a^2(27a^2+1)}\tau, v_2 \simeq -\frac{8d^2 h_I (b-d)}{9a^2(27a^2+1)}\tau,$$

and return the solution to the original variables, we acquire

$$x \simeq -\frac{2dh_I}{3a(27a^2+1)}t, y \simeq \frac{8d^2 h_I (b-d)}{9a^2(27a^2+1)}t, z \simeq -\frac{8d^2 h_I (b-d)}{9a^2(27a^2+1)}t, I \simeq -\frac{8d^2 h_I (b-d)}{9a^2(27a^2+1)}t, \quad (30)$$

**Remark 3.1** It is evident that the canard solutions for the original system, simultaneously occurring at time t in equations (25) and also in equations (30) which are infinitely close result. At the pseudo singular points, using infinitesimal changes of coordinates provides the solution for the local model (1).

## 4. Conclusion

This work presents an expanded version of the three-dimensional Hindmarsh-Rose model. Under the conditions found for equations (17) and (23), the case of pseudo singular points of saddle-type has been analyzed. The existence of canard solutions in the four-dimensional Hindmarsh-Rose model could be proven by applying a nonstandard technique developed in the reference (Tchizawa, 2013).

## References


BENOÎT, É. 1983. Systemes lents-rapides dans $R^3$ et leurs canards. *Astérisque*, 109-110.

BENOÎT, É. & LOBRY, C. 1982. Les canards de $R^3$. *CR Acad. Sc. Paris,* 294, 483-488.

CORSON, N. & AZIZ-ALAOUI, M. 2009. Asymptotic dynamics of Hindmarsh-Rose neuronal system. *Dynamics of Continuous, Discrete and Impulsive Systemes, Series B: Applications and Algorithms*, p. 535.

DESROCHES, M., KAPER, T. J. & KRUPA, M. 2013. Mixed-mode bursting oscillations: Dynamics created by a slow passage through spike-adding canard explosion in a square-wave burster. *Chaos: An Interdisciplinary Journal of Nonlinear Science,* 23.

DUMORTIER, F. & ROUSSARIE, R. H. 1996. *Canard cycles and center manifolds*, American Mathematical Soc.

GINOUX, J.-M. & LLIBRE, J. 2016. Canards existence in memristor's circuits. *Qualitative theory of dynamical systems,* 15, 383-431.

GINOUX, J.-M., LLIBRE, J. & TCHIZAWA, K. 2019. Canards existence in the Hindmarsh–Rose model. *Mathematical Modelling of Natural Phenomena,* 14, 409.

HAMAD, I., QADIR, C. & AZIZ, W. 2023. Nonstandard canard solutions in a three-dimensional system and Hindmarsh-Rose neuron model. Submitted.

SZMOLYAN, P. & WECHSELBERGER, M. 2001. Canards in R3. *Journal of Differential Equations,* 177, 419-453.

TCHIZAWA, K. 2007. Generic conditions for duck solutions in $R^4$ *Kyoto Univ RIMS Kokyuroku,* 1547, 107-113.

TCHIZAWA, K. 2010. On 4-dim duck solutions with relative stability (Dynamical Systems: with Hyperbolicity and with Large Freedom). *数理解析研究所講究録,* 1688, 157-163.

TCHIZAWA, K. 2012. On relative stability in 4-dimensional duck solution. *Journal of Mathematics and System Science,* 2, 558.

TCHIZAWA, K. 2013. On the two methods for finding 4-dimensional duck solutions. *Applied Mathematics,* 2014.

TCHIZAWA, K., MIKI, H. & NISHINO, H. 2005. On the existence of a duck solution in Goodwin's nonlinear business cycle model. *Nonlinear Analysis: Theory, Methods & Applications,* 63, e2553-e2558.

VAN DER POL, B. 1926. On "relaxation-oscillations". *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science,* 2, 978-992.

VAN DER POL, B. 1960. A theory of the amplitude of free and forced triode vibrations, Radio Rev. 1 (1920) 701-710, 754-762. *Selected scientific papers,* 1.

WECHSELBERGER, M. 2012. A propos de canards (apropos canards). *Transactions of the American Mathematical Society,* 364, 3289-3309.