RESEARCH PAPER

Purely Semismall Compressible Modules

Mukdad Qaess Hussain¹,a,*
Zahraa Jawad Kadhim²,b
Shahad Jasim Mahmood³,c

¹College of Education for pure science, Diyala University
²Computer Engineering, Al-mansur University College
³College of Science, Diyala University

*a mukdazaess2016@yahoo.com
*b zahraa.jawad@muc.edu.iq
*c shahADjasim@uodiylala.edu.iq

ABSTRACT:
Let R be a ring with 1 and D be unitary left Module over R. In this paper, we present purely semi small compressible Modules. Also, we give remarks and examples, many properties of such Modules are investigated.

KEY WORDS: Semi small Submodule, prime Modules, pure Modules, semi small compressible Module.
DOI: http://dx.doi.org/10.21271/ZJPAS.34.s6.2
ZJPAS (2022), 34(s6):9-11.

1. INTRODUCTION:
A Submodule W of an R-Module D is small Submodule (W ≪ D) if W + U = D for any Submodule U of D then U = D[1]. A proper Submodule W of an R-Module D is semismall of D (W ≪₅ D) if W = 0 or W/U ≪ D/U ∀ nonzero Submodule U of W [2]. An R-Module D is semismall compressible (S.S.C.) if D embedded in all its non-zero semismall Submodule. That is, D is S.S.C. if ∃ a monomorphism f: D → W whenever 0 ≠ W ≪₅ D[3]. A ring R is S.S.C. if R as R-Module is S.S.C. [3]. In this paper, we give purely semismall compressible Modules as a generalization of semismall compressible Modules.


For example:
- Z₄ as a Z-Module is P.S.S.C. which is not S.S.C.
- An R-Module D is purely semismall simple if pure semismall Submodules of D are (0) and D.
- Every purely semismall simple Module is P.S.S.C.
- Every integral domain R is a P.S.S.C. R-Module.
- A ring R is regular ring if for every r ∈ R ∃ w ∈ R such that r = rwr[4].
- An R-Module D is regular Module if for every u ∈ D and ∀ r ∈ R, ∃ w ∈ R such that ru = rwr[5].
- Every regular Module D is S.S.C. iff D is P.S.S.C.
- Every Module D over regular ring R is S.S.C. iff D is P.S.S.C.

* Corresponding Author: Mukdad Qaess Hussain
E-mail: mukdadqaess2016@yahoo.com
Article History:
Received: 01/08/2022
Accepted: 15/09/2022
Published: 22/7/12/2022
Proposition 1. Let $D_1$ and $D_2$ be two isomorphic Modules. Then $D_1$ is P.S.S.C. iff $D_2$ is P.S.S.C.

Proof: Assume $D_1$ is P.S.S.C. and an isomorphism $\varphi: D_1 \to D_2$. Let $0 \neq W$ be a pure semismall Submodule of $D_2$. Put $K = \varphi^{-1}(W)$, thus $K$ is a semismall submodule of $D_1$. Claim $K$ is pure in $D_1$. Let $j$ be an ideal of $R$. But $f$ is a monomorphism gives $\varphi((D_1 \cap K) = \varphi((D_2 \cap \varphi^{-1}(W)) = jD_2 \cap W = jW = j\varphi(K) = \varphi(jK)$. But $\varphi$ is an isomorphism, then $D_1 \cap K = jK$. Hence $K$ is pure in $D_1$. Let $f: D_1 \to K$ be a monomorphism and let $g = \varphi|_K$ thus $g: K \to D_2$ is a monomorphism and $g(K) = \varphi(\varphi^{-1}(W)) = W$, hence $g: K \to W$ is a monomorphism. Now, we have the composition $D_2 \xrightarrow{\varphi^{-1}} D_1 \xrightarrow{f} K \xrightarrow{g} W$. Let $Y = gf\varphi^{-1}$ is a monomorphism. Thus $D_2$ is P.S.S.C.

Proposition 2. Every non-zero pure semismall Submodule of a P.S.S.C. Module is P.S.S.C.

Proof: Assume $W$ be a non-zero pure semismall Submodule of P.S.S.C. Module $D$. Assume $U$ be a pure semismall Submodule of $W$. Thus, by [6, Remarks 1.2.8] $U$ is pure in $D$. Then there is a monomorphism $f: W \to U$ and hence if: $U \to W$ is a monomorphism where i: $U \to W$ be inclusion homomorphism. Thus $W$ is P.S.S.C.

Corollary 3. Every direct summand of P.S.S.C. Module is P.S.S.C.


Corollary 5. Let $R$ be a regular ring. Every non-zero semismall Submodule of P.S.S.C. is P.S.S.C.

Remark 6. A homomorphic image (quotient) of P.S.S.C. Module is not always be P.S.S.C. For example, $Z$ as $Z$-Module is P.S.S.C., but $Z/6Z \cong Z_6$ is not a P.S.S.C. $Z$-Module.

Remark 7. The direct sum of P.S.S.C. Modules is not P.S.S.C.

Example 8. Let $D = Z_4 \oplus Z_2$ as a $Z$-Module. Clearly $Z_4$ and $Z_2$ is P.S.S.C. $Z$-Module. Since $D$ is not P.S.S.C. But $D = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\}$. Thus, $F = Z(1, 1) = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$, and $G = Z(2, 1) = \{(0, 0), (2, 1)\}$. Clearly, $D = F \oplus G$. Thus, $F$ and $G$ is a pure semismall Submodule of $D$, since $D$ cannot be embedded in $F$ (or in $G$). Thus $D$ is not P.S.S.C.

Example 9. Let $D = Z_6$ as a $Z$-Module and $W = \langle 2 \rangle$. $W$ is pure semismall in $Z_6$ and $W$ is P.S.S.P. Submodule of $D$.

Lemma 10. An $R$-Module $D$ is P.S.S.P. iff $(0)$ is a P.S.S.P. Submodule of $D$.

Proof: $(\Rightarrow)$ Suppose $wz = 0$ with $w \in R, z \in D$ and $(z)$ is pure in $D$. Assume $z \neq 0$. Since $D$ is P.S.S.P. then $\text{ann}(D) = \text{ann}(z)$ and hence $w \in \text{ann}(D) = \{0: D\}$. Thus $(0)$ is a P.S.S.P. Submodule of $D$.

$(\Leftarrow)$ Assume $(0)$ is a P.S.S.P. Submodule of $D$ and $0 \neq W$ be a pure semismall Submodule of $D$. Let $w \in \text{ann}(W)$. Thus $wz = 0$ for all $z \in W$, and hence $wz \in (0)$. Assume that $z \neq 0$, then $w \in [0: D] = \text{ann}(D)$, thus $\text{ann}(W) \subseteq \text{ann}(D)$, so $\text{ann}(D) = \text{ann}(W)$, then $D$ is P.S.S.P.

Lemma 11. Let $D$ be a Module and every Submodule of a pure Module is pure. If $W$ is P.S.S.P. Module, then $\text{ann}(W)$ is a pure semismall ideal of $R$ for each non-zero pure semismall Submodule $W$ of $D$. 
Proof: Assume $0 \neq W$ be a pure Submodule of $D$. let $a,b \in R$ and $ab \in \text{ann} W$. Thus $abW = 0$. Suppose that $bW \neq 0$. Since $bW \leq W$ and $W$ is pure in $D$ then $bW$ is pure in $D$. Since $D$ is P.S.S.P. and $a \in \text{ann} bW$, then $a \in \text{ann} D$, on the other hand $\text{ann} D = \text{ann} W$, so $a \in \text{ann} W$ and hence $\text{ann} W$ is a semismall prime ideal of $R$. The converse of Lemma 11 is not true.

Example 12. $Z_6$ is not P.S.S.P. Z-Module, however $\text{ann}_Z(2) = 3Z$ and $\text{ann}_Z(3) = 2Z$ which are both prime ideals in $Z$ and that $(2), (3)$ are pure Submodule of $Z_6$.

Proposition 13. Every P.S.S.C. Module is P.S.S.P.
Proof: Assume $D$ be P.S.S.C. Module and $0 \neq W$ be a pure semismall Submodule of $D$. To show $\text{ann} D = \text{ann} W$. Let $x \in \text{ann} W$. Thus $xW = 0$. Let $f : D \rightarrow W$ be a monomorphism, thus $f(xD) = xf(D) \subseteq xW = 0$ implies that $xD = 0$, then $x \in \text{ann} W$ and hence $\text{ann} D = \text{ann} W$.

2. CONCLUSIONS
In this paper, we give purely semismall compressible Modules. Also, we give remarks and examples, many properties of such Modules

- $D_1$ is purely semismall compressible. iff $D_2$ is purely semismall compressible, where $D_1$ and $D_2$ be two isomorphic Modules.
- Every non-zero pure semismall Submodule of purely semismall compressible Module is purely semismall compressible.
- Every direct summand of purely semismall compressible Module is purely semismall compressible.
- Every non-zero semismall Submodule of regular purely semismall compressible Module is purely semismall compressible.
- Every non-zero semismall Submodule of purely semismall compressible in regular ring is purely semismall compressible.
- A homomorphic image (quotient) of purely semismall compressible Module is not always be purely semismall compressible.
- The direct sum of purely semismall compressible Modules is not purely semismall compressible.
- Every purely semismall compressible Module is purely semismall prime.

References